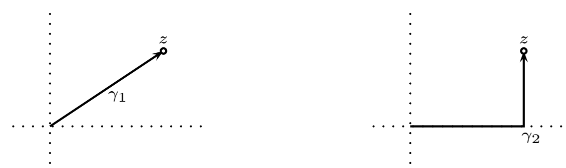
Math 326 — Homework Assignment 5

DUE DATE: OCT. 21, 2010

1. Let $z_0 \in \mathbb{C}$ be a fixed complex number, and for any real number $r > 0$ let γ_r be the circle of radius r around z_0, oriented counterclockwise. Find $\displaystyle\int_{\gamma_r} \frac{1}{z - z_0}\, dz$.

2. For a point $z = x + iy \in \mathbb{C}$ let γ_1 be the path that goes in a straight line from 0 to z, and let γ_2 be the path that goes from 0 to x, and then to z, as illustrated below.



Let $f(z) = 3|z|^2$.

(a) Find $\displaystyle\int_{\gamma_1} f(z)\, dz$.

(b) Find $\displaystyle\int_{\gamma_2} f(z)\, dz$.

The answers in (a) and (b) depend on $z = x + iy$ and so define complex functions $F_1(z)$ and $F_2(z)$.

(c) Is the function F_1 from (a) holomorphic?

(d) Is the function F_2 from (b) holomorphic?

3. Show the following estimates:

(a) $\displaystyle\left|\int_\gamma \frac{dz}{z^2 - i}\right| \le \frac{3\pi}{4}$, where $\gamma = \Big\{ z \in \mathbb{C} \,\Big|\, |z| = 3 \Big\}$, oriented counterclockwise.

(b) $\displaystyle\left|\int_\gamma \operatorname{Log}(z)\, dz\right| \le \frac{\pi^2}{4}$, where $\gamma = \Big\{ z \in \mathbb{C} \,\Big|\, |z| = 1,\ 0 \le \operatorname{Arg}(z) \le \tfrac{\pi}{2} \Big\}$, oriented counterclockwise.

(c) $\displaystyle\left|\int_\gamma \exp(\sin z)\, dz\right| \le 1$, where γ is the straight line segment from $z = 0$ to $z = i$.

4. The definition of the complex integral along a contour was obtained by imitating the real definition via Riemann sums. It would be nice if there were also a simple geometric description of what the complex integral is measuring. The purpose of this problem is to give one possible geometric description of $\int_\gamma f(z)\,dz$, although it is terms of the $\mathbb{R}^2$ vector field associated to $\overline{f(z)}$ instead of the vector field associated to $f(z)$.

Suppose that $\mathbf{G} = (G_1, G_2)\colon S \longrightarrow \mathbb{R}^2$ is function defined on an open set S in $\mathbb{R}^2$, and that $\gamma\colon (a,b) \longrightarrow \mathbb{R}^2$ is a parametrized curve given by $\gamma(t) = (x(t), y(t))$ whose image lies in S. Recall that the flow of $\mathbf{G}$ along γ (the usual line integral) is given by

$$\int_\gamma \mathbf{G} \cdot ds = \int_a^b G_1(\gamma(t))x'(t) + G_2(\gamma(t))y'(t)\,dt$$

and that the flow of $\mathbf{G}$ through γ, from right to left, is given by the integral

$$\int_a^b G_2(\gamma(t))x'(t) - G_1(\gamma(t))y'(t)\,dt.$$

(The reason for the choice from right to left is that if γ is a closed curve, oriented counterclockwise, then the flow through γ is the same as the flow through the region enclosed by γ).

Now suppose that $f(z)\colon S \longrightarrow \mathbb{C}$ is a function defined on an open set S, and γ is a curve in S. Let $\mathbf{G}$ be the function $S \longrightarrow \mathbb{R}^2$ associated to $\overline{f(z)}$ (by taking real and imaginary parts as usual).

Show that

$$\operatorname{Re}\left(\int_\gamma f(z)\,dz\right) = \text{flow of } \mathbf{G} \text{ along } \gamma$$

and

$$\operatorname{Im}\left(\int_\gamma f(z)\,dz\right) = -(\text{flow of } \mathbf{G} \text{ through } \gamma).$$

NOTE: Despite the long discussion needed to set up this problem, the solution is quite short, so don't lose hope.