DUE DATE: Nov. 4, 2010

1. Let f be an entire function, $z_0 \in \mathbb{C}$ a fixed point, and γ a circle of some radius around z_0. Assuming:

(i) Cauchy's integral theorem, i.e., that

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w - z}\, dw$$

for all z inside of γ, and

(ii) that we can exchange differentiation and integration, (i.e., that we can differentiate under the integral sign),

use induction to prove Cauchy's integral formula for derivatives:

$$f^{(k)}(z) = \frac{k!}{2\pi i} \int_\gamma \frac{f(w)}{(w - z)^{k+1}}\, dw$$

for all z inside of γ and all $k \geq 0$.

2. Use Cauchy's integral theorem (and the version for derivatives) to solve each of the following integrals without explicitly integrating.

(a) $\displaystyle \frac{1}{2\pi i} \int_{|z|=1} \frac{\exp(3z)}{z^2}\, dz$ (b) $\displaystyle \int_{|z|=3} \frac{\sin(z)}{z^4} + \frac{\exp(z)}{z-2}\, dz$

(c) $\displaystyle \int_{|z|=2} \frac{1}{z(z-5)^2}\, dz$ (d) $\displaystyle \int_{|z-4|=2} \frac{1}{z(z-5)^2}\, dz$

NOTE: Leading factors (like $\frac{k!}{2\pi i}$) may have to be adjusted to make the integrals above fit the exact statement of the integral theorems. All contours should be taken counter-clockwise.

3. Let n be a positive integer. We say that a function f has *polynomial growth of order at most* n if there are constants $c, R_0 \in \mathbb{R}$ such that $|f(z)| \leq c|z|^n$ for all $z \in \mathbb{C}$, $|z| \geq R_0$. Show that if f is an entire function of polynomial growth of order at most n then f is a polynomial of degree at most n.

NOTE: The case $n = 0$ is Liouville's theorem, and perhaps the proof of that theorem can be modified to cover the more general version above.

4. In order to get a better feel for what the theorems about "integrals of Cauchy type" are saying, it would be helpful to actually work out an example of a function defined by such an integral. Let γ be the straight line from -1 to 1, and let g be the constant function $g(w) = 1$ for all $w \in [-1, 1]$. By integrating, work out an explicit formula for the function

$$G(z) = \frac{1}{2\pi i} \int_\gamma \frac{g(w)}{w - z}\, dw,$$

Since you will probably want to split the integral into its real and imaginary parts, it might be helpful to write $z = x + iy$ (or $z = a + bi$) to make this easier. You might also want to consider the cases $y \neq 0$ and $y = 0$ separately. Note that G is defined on $\mathbb{C} \setminus [-1, 1]$.

5. Suppose that f is an entire function, and let γ be the circle of radius 8 around $z = 0$. Define a function $G(z)$ by

$$G(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w - z}\, dw,$$

for all $z \notin \gamma$.

(a) Describe $G(z)$ inside γ.

(b) Describe $G(z)$ outside γ.

Note: The answers to parts (a) and (b) are extremely short. The purpose of this question is just to think again about the definition of such functions, and to think about what Cauchy's theorems are telling us.