DUE DATE: NOV. 11, 2010

1. Suppose that $p(z)$ is a polynomial with $\deg p \geq 2$, and let $f(z) = \frac{1}{p(z)}$.

   (a) Explain why there is an $R_0 \in \mathbb{R}$ so that $f(z)$ is holomorphic on the set $\left\{ z \,\middle|\, |z| > R_0 \right\}$.

   (b) Explain why, for every $R > R_0$, $\int_{|z|=R} f(z)\, dz = \int_{|z|=R_0} f(z)\, dz$.

   (c) Estimate $|f(z)|$ on the circle $|z| = R$ (possibly just for large R) and use the estimate to show that $\int_{|z|=R_0} f(z)\, dz = 0$.

2.

   (a) Give an example of a domain D and a holomorphic function $f(z)$ such that $|f(z)|$ achieves its minimum inside D, but $f(z)$ isn't constant. (I.e., without further hypotheses, there isn't a "minimum modulus theorem" for holomorphic functions).

   (b) Suppose that D is a domain, and f a function holomorphic on D which is never zero on D. Show that if $z_0 \in D$ is a local minimum for $|f(z)|$ then f is constant near z_0.

SUGGESTION: Don't try and prove (b) from scratch. Instead try to reduce it to a result we already know.

3. Prove a Liouville-type theorem for harmonic functions: If u is function harmonic on all of $\mathbb{R}^2$, then if u is bounded above or bounded below, u must be constant.

HINT: Suppose that u is a harmonic function defined on all of $\mathbb{R}^2$ and that v is its harmonic conjugate. Let $f = u + iv$ be the corresponding entire function. What is $|\exp(f(z))|$?

4.

   (a) Show that $\dfrac{d}{d\theta} 2\arctan\left(\dfrac{1+r}{1-r} \cdot \tan\left(\dfrac{\theta}{2}\right)\right) = \dfrac{1-r^2}{1-2r\cos(\theta)+r^2}$.

   (b) Find a function $u(x,y)$ harmonic inside the unit disc $|z| \leq 1$ such that $u = 1$ on the top half of the unit circle, and $u = 0$ on the bottom half of the unit circle. I.e., solve the Dirichlet problem for the disk with the boundary conditions that the function be 1 on the top of the circle and 0 on the bottom of the circle.

NOTE: In part (b), you should use Poisson's formula rather than try and pull an answer out of thin air. You do not have to verify that the resulting function is harmonic.

5.

(a) For positive real numbers R and r, and any angle θ, show that

$$(R-r)^2 \leq R^2 - 2Rr\cos(\theta) + r^2 \leq (R+r)^2.$$

(b) If u is a function harmonic on the disk of radius R and nonnegative on the boundary, use Poisson's formula, the mean value formula for harmonic functions, and part (a) to establish *Harnack's inequality:*

$$\frac{R-r}{R+r}u(0) \leq u(r\exp(i\alpha)) \leq \frac{R+r}{R-r}u(0)$$

for any $0 < r < R$ and any angle α.