Math 326 — Homework Assignment 9

DUE DATE: Nov. 18, 2010

1. In class we used Cauchy's integral formula and Cauchy's integral formula for derivatives to deduce the existence of the Taylor series expansion for a holomorphic function $f(z)$ around a point z_0. The point of this question is to reverse that implication and show that, conversely, if we knew that each function has a Taylor expansion then we can deduce Cauchy's integral formula for derivatives (which includes Cauchy's integral formula).

Suppose that f is holomorphic in the disk $D_r(z_0) = \left\{ z \,\middle|\, |z - z_0| < r \right\}$ and has Taylor series expansion

$$f(z) = \sum_{n \geq 0} a_n (z - z_0)^n$$

in $D_r(z_0)$.

(a) What is the formula for the coefficients a_n in terms of the derivatives of f at z_0?

(b) Suppose that γ is a circle around z_0 of radius less than r (i.e., γ is contained in $D_r(z_0)$ so that the expansion above is valid on γ). Compute

$$\frac{n!}{2\pi i} \int_\gamma \frac{f(z)}{(z - z_0)^{n+1}}\, dz$$

by using the power series expansion for f and exchanging integration and summation. (Question 2 on Homework 6 may also be useful).

2. In class we worked out the constant term (the z^0 term) of the Laurent series for $\exp(\frac{1}{z} + z)$ by multiplying the series for $\exp(\frac{1}{z})$ and $\exp(z)$.

(a) By substituting $w = z + \frac{1}{z}$ into the series for e^w and expanding, compute the constant term and see that it agrees with the answer found in class.

(b) Find a formula for the z^n term of the Laurent series for $e^{\frac{1}{z} + z}$ (by either method); here $n \in \mathbb{Z}$ (i.e, n could be negative).

3. The Taylor series expansion

$$\frac{1}{1 - z} = \sum_{n \geq 0} z^n = 1 + z + z^2 + z^3 + \cdots$$

valid for $|z| < 1$ is extremely useful in analysis (and combinatorics, and many other places). By differentiating this series, find the formula for the Taylor series expansion of $\frac{1}{(1-z)^k}$. You should be able to write the coefficient of z^n in the expansion as a binomial coefficient, and this is the cleanest way to write the formula.

4.

(a) Find the Laurent series expansion for $\frac{1}{1+z^2}$ valid on the annulus $\left\{z \,\middle|\, |z| > 1\right\}$.

HINT: Rewrite $\frac{1}{1+z^2}$ in such a way so that you can essentially expand it as a geometric series with ratio $-\frac{1}{z^2}$.

(b) By integrating the answer from (a) find a Laurent series valid on $|z| > 1$ which agrees with arctan on the positive real axis. (The only thing not determined by the integration is the constant term. However, as $z \to \infty$ on the positive real axis, you know how arctan behaves, and this should determine the constant).

(c) Use your answer from (b), the fact that $\arctan(\sqrt{3}) = \frac{\pi}{3}$ and that $\sqrt{3} > 1$ to find an infinite series for $\frac{\pi}{6}$. (The switch from $\frac{\pi}{3}$ to $\frac{\pi}{6}$ comes by combining the answer with the constant term.) Multiply this answer by 6 to find an infinite series for π.

NOTE: I know of no good use for the series expression in (c), but it is a formula for π that you probably haven't seen before.

5. Let $f(z) = \sum_{n=-\infty}^{\infty} z^n$. The purpose of this question is to show that f is the zero function in two different ways, and then show why both of these arguments are wrong.

(a) Multiply the series for f by $z-1$ and rewrite as a Laurent series (i.e., collect powers of z) to show that $(z-1) \cdot f(z) = 0$. Since $z-1$ is zero only at 1 this means that $f(z)$ must be zero at all other $z \in \mathbb{C}$. By continuity this means that f must be zero at 1 as well.

(b) Using the formula for a geometric series, show that $\frac{1}{1-z} = 1+z+z^2+\cdots$. Similarly, writing $\frac{1}{z-1}$ as $\frac{1}{z} \cdot \frac{1}{1-\frac{1}{z}}$, show that $\frac{1}{z-1}$ can be expanded as a series in $\frac{1}{z}$. Add these two answers to conclude again that $f(z) = 0$.

(c) The conclusions in (a) and (b) say that f is the zero function. But Laurent expansions are supposed to be unique, and surely the expansion of the zero function is just $0 = \sum_{n=-\infty}^{\infty} 0 \cdot z^n$, so this seems to be a contradiction. Explain what went wrong in parts (a) and (b).