1.

(a) Suppose that $p(z) = (z - \alpha_1)(z - \alpha_2) \cdots (z - \alpha_n)$ is a polynomial of degree n with distinct roots $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$. Let $f(z) = \frac{1}{p(z)}$. Find a formula for $\mathrm{Res}(f; \alpha_k)$ (where $k \in \{1, \ldots, n\}$).

(b) Suppose that $p(z)$ is a polynomial of degree n, and let $\alpha_1, \ldots, \alpha_m$ be the roots of p (m might not equal n since some of the roots may be repeated). Let $f(z) = \frac{1}{p(z)}$. Explain why there is real number R so that

$$\int_{|z|=R} f(z)\, dz = 2\pi i \sum_{k=1}^{m} \mathrm{Res}(f; \alpha_k).$$

(c) If $n \geq 2$, explain why $\sum_{k=1}^{m} \mathrm{Res}(f; \alpha_k) = 0$. (A previous homework assignment may be helpful here). Is this still true if $n = 1$?

(d) Suppose again that $p(z)$ has distinct roots, and that $n \geq 2$. What formula results by combining the answers to (a) and (c)?

2. Suppose that $f(z)$ has a pole of order k at z_0. The purpose of this question is to understand short-cut 5 for computing the residue at z_0.

(a) What does the Laurent series for $f(z)$ look like around z_0?

(b) What does the Laurent series for $(z - z_0)^k f(z)$ look like around z_0?

(c) Let $\varphi(z)$ be the function defined by the series in (b). What is the series for $\varphi^{(k-1)}$, the $(k-1)$-st derivative of φ?

(d) What is the relation between the constant term for the series in (c) and $\mathrm{Res}(f; z_0)$?

(e) In short-cut 5, why do we take the limit as $z \to z_0$, instead of just "plugging in $z = z_0$"?

3. For each of the following functions, find the singular points and compute the residues at those points. (NOTE: the result of 1(c) should provide a check for your answers).

(a) $\dfrac{1}{z^3(z+4)}$ (b) $\dfrac{1}{z^2 + 2z + 1}$ (c) $\dfrac{1}{z^2 - 3}$

4. Suppose that f_1 and f_2 have simple poles at z_0. By writing out the Laurent expansions, show that $f_1 f_2$ has a pole of order 2 at z_0, and find a formula for $\mathrm{Res}(f_1 f_2; z_0)$ (in terms of the coefficients of the Laurent expansions of f_1 and f_2).