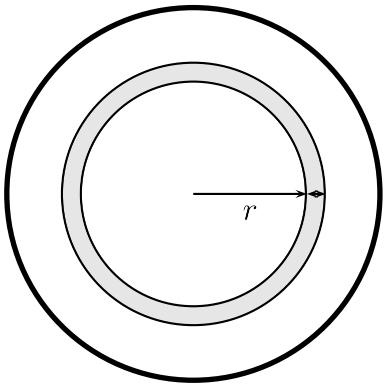
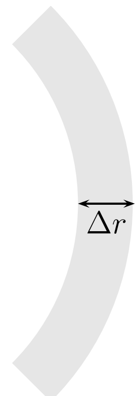
Approximations in setting up Riemann Sums

1. The area of a ring.

Last class, the issue came up of how to approximate the area of a circular ring, with inner radius r and outer radius $r + \Delta r$.



Circular ring



Detail of ring

The exact area is

$$\pi(r + \Delta r)^2 - \pi r^2 = \pi(r^2 + 2r\,\Delta r + (\Delta r)^2 - r^2) = 2\pi r\,\Delta r + \pi(\Delta r)^2.$$

There were some people who felt that $2\pi r\,\Delta r$ would do as well, even though it isn't the exact formula for the area (it's missing the $\pi(\Delta r)^2$ term).

In fact, that's correct – when setting up the Riemann sum you can simply ignore the $\pi(\Delta r)^2$ part.

How can using an incorrect formula give you a correct answer? Read on to find out.

2. Δx versus $(\Delta x)^2$.

Suppose we're trying to compute a Riemann sum over the interval $[0, 10]$, adding up some formula that involves both Δx and $(\Delta x)^2$. (It could be the formula above with x in place of r for example.)

Our plan, as always for Riemann sums, is to divide the interval up into n pieces, evaluate the function on the pieces, and then add up the values on the pieces.

That means we'll be adding up n terms. We'll see a term involving Δx appear n times in the sum, and a term involving $(\Delta x)^2$ appear n times in the sum. As we increase n, both Δx and $(\Delta x)^2$ decrease, but since we'll be adding them up n times, it might not be so clear what will happen.

Here's a little table for various values of n, of n, Δx, $(\Delta x)^2$, $n \cdot \Delta z$, and $n \cdot (\Delta x)^2$:

n	Δx	$(\Delta x)^2$	$n \cdot \Delta x$	$n \cdot (\Delta x)^2$
1	10	100	10	100
10	1	1	10	10
100	0.1	0.01	10	0.1
1000	0.01	0.0001	10	0.01
10000	0.001	0.000001	10	0.001
100000	0.0001	0.00000001	10	0.0001

(Note: the interval is $[0, 10]$, so $\Delta x = \frac{10}{n}$.)

You can see that as n increases, the contribution from the $n \cdot \Delta x$ term remains constant, but the contribution from the $n \cdot (\Delta x)^2$ term is disappearing. That seems to imply that its contribution to the sum will also disappear as n gets large.

Let's try this out with a concrete example involving density on a circular shape.

3. Two computations of the mass.

Just to pick a concrete example, let's suppose that we have a circular shape of radius 10, and that the density of this shape at a distance r from the center is given by

$$\rho(r) = \sqrt{100 - r^2}.$$

What is the total mass of the circular shape?

If we divide the circle up into n concentric rings, the mass of each ring is either

$$\sqrt{100 - r^2} \cdot (2\pi r\, \Delta r + \pi(\Delta r)^2)$$

using the correct expression for the area, or

$$\sqrt{100 - r^2} \cdot 2\pi r\, \Delta r$$

using the approximate expression for the area.

That means we now have two possible candidates for a sum which approximates the mass:

$$\sum_{i=1}^{n} (\sqrt{100 - r_i^2}) \cdot (2\pi r_i\, \Delta r + \pi(\Delta r)^2) \quad \text{or} \quad \sum_{i=1}^{n} (\sqrt{100 - r_i^2}) \cdot 2\pi r_i\, \Delta r.$$

Let's compare the results given by these sums as n gets large.

n	$\sum_{i=1}^{n}(\sqrt{100-r_i^2})\cdot(2\pi r_i\,\Delta r+\pi(\Delta r)^2)$	$\sum_{i=1}^{n}(\sqrt{100-r_i^2})\cdot 2\pi r_i\,\Delta r$
1	3141.592655	0.000000
10	2291.178682	2031.642419
100	2117.320204	2092.498347
10000	2094.640009	2094.393259
100000	2094.419706	2094.395049
1000000	2094.397755	2094.395288

As you can see, as n gets large, the sum without the $(\Delta r)^2$ term is giving the same answer as the sum with the $(\Delta r)^2$ term.

In other words: The contribution of the $(\Delta r)^2$ term makes no difference in the limit, and hence no difference to the final answer.

That's a powerful trick: When setting up our Riemann sums, we can ignore terms which have $(\Delta r)^2$, or in general, powers of (Δx) higher than one.

This handout can (soon) be found at

http://www.mast.queensu.ca/∼mikeroth/calculus/calculus.html

E-mail address: mikeroth@mast.queensu.ca