Equations in Mathematics

1. The golden rectangle.

The Greeks (among other people) attached mystic and esthetic meaning to many mathematical objects.

As an example, there was one rectangle that they considered more beautiful than any other rectangle, one whose proportions were the most harmonious, and which was visually the most pleasing. They called this rectangle the *golden rectangle*.

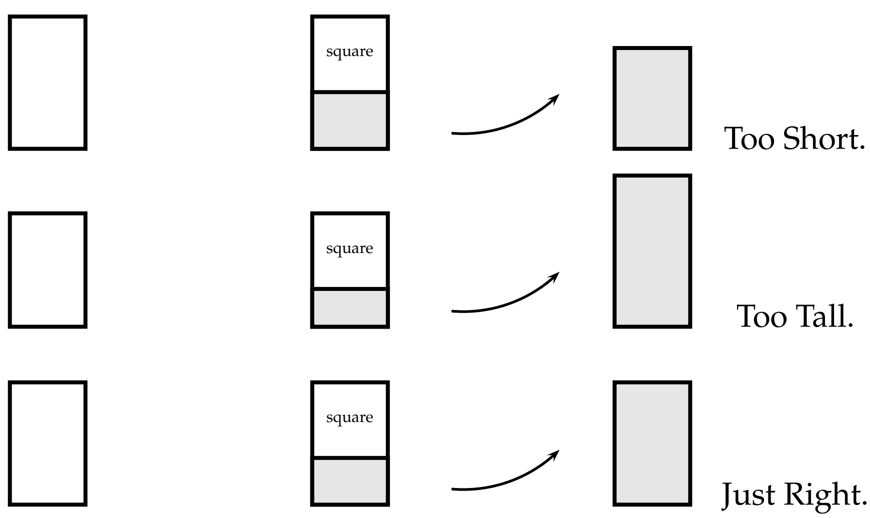
The golden rectangle shows up in many Greek buildings, including the Parthenon, where it appears everywhere. During the Renaissance, when European artists and scientists were rediscovering the ideas of antiquity, the Greek ideas of proportion were also picked up, and the golden rectangle found its way into Renaissance art, influencing the composition of paintings.

What distinguishes the golden rectangle among other rectangles?

First of all, to the Greeks (and the artists) it was only the proportions of the sides of the rectangle to each other that mattered, not the absolute length of the sides. In other words, if you take a rectangle and scale the sides by the same factor (say, multiplying by 2), that would be considered the same rectangle.

Each rectangle has a short side and a long side (we're ignoring squares, in this case we don't consider them to be rectangles). Given any rectangle, consider the largest possible square sitting inside the rectangle, it will be a square with side length the same as the length of the shortest side of the rectangle.

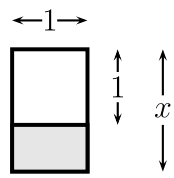
The part outside the square is another rectangle. The golden rectangle is the rectangle such that, when you cut out the square, the rectangle you're left with is the same as the rectangle you started with.



2. The dimensions of the golden rectangle.

What are the dimensions of the golden rectangle?

Since the size of the rectangle only matters up to scaling, we can assume that the short side of the rectangle is 1. Let's use x for the length of the long side of the rectangle. Here's what we get after we cut the square out of the rectangle:



The fact that this is a golden rectangle gives us the equation

$$\frac{x}{1} = \frac{1}{x-1}.$$

Multiplying, we get $x^2 - x = 1$ or $x^2 - x + 1 = 0$, and this is something we can easily solve with the quadratic formula:

$$x = \frac{1 \pm \sqrt{5}}{2}.$$

Since we're looking for a solution with $x > 1$ (x was the long side), we see that the solution has to be

$$x = \frac{1 + \sqrt{5}}{2}.$$

3. Equations in mathematics.

Finding the dimensions of the golden ratio was an example of a very common procedure. If we're trying to solve a problem which has a number as an answer, we usually

– Pick a name (like x) to stand for the unknown number.

– Use the details of the problem to write down an equation involving x.

– Solve the equation to find x, and so find our number.

In calculus, we move from asking questions about numbers to asking questions about functions.

We'll be looking at problems where the answer is a function. In order to solve the problem, we'll go through the same series of steps:

- Pick a name (like $f(t)$, or $y(t)$) to stand for the unknown function.
- Use the details of the problem to write down an equation involving f (or y).
- Solve the equation to find f, and so find our function.

The new twist is this: there are operations that are possible with functions which aren't possible with numbers. For instance, functions can have derivatives, and so one thing that might happen is that our equation might involve not only our unknown function f, but also its derivatives.

Here's an example:

$$f''(t) - 3f'(t) + 2f(t) = \cos(t),$$

or, in y notation:

$$y''(t) - 3y'(t) + 2y(t) = \cos(t),$$

or

$$\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y = \cos(t).$$

Equations like this, involving a function and its derivatives, are called *differential equations*.

Our goal is to learn to do the steps above in simple situations. We'll want to be able to look at a problem and write down the differential equation satisfied by the unknown function, and we'll want to know how to solve the differential equation to find that mystery function.

This handout can (soon) be found at

http://www.mast.queensu.ca/∼mikeroth/calculus/calculus.html

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