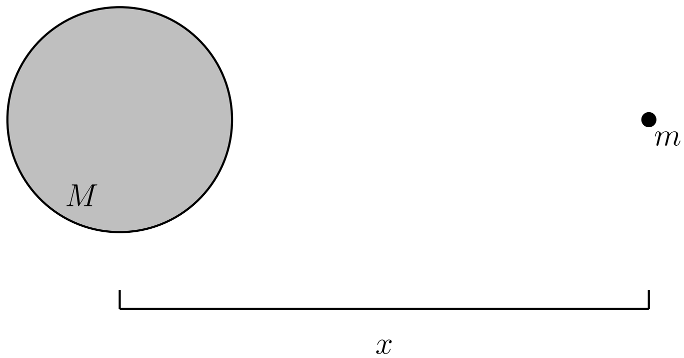
An example from Physics

1. Gravitational Force.

Suppose that we have a planet of mass M and that at distance x from the planet there is another object of mass m.



Newton's law of gravitation says that the gravitational force between the planet and the object is given by the formula

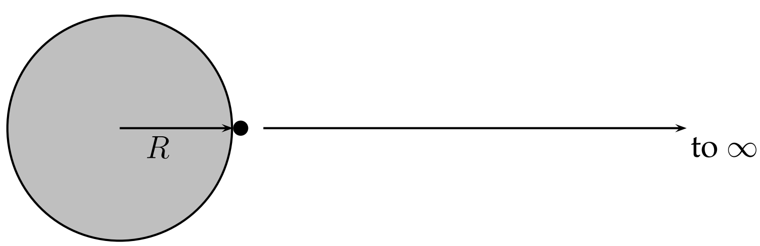
$$F = \frac{GMm}{x^2},$$

where G is the universal gravitational constant. If m and M are measured in kilograms, x in metres, and the force F in Newtons, then then the gravitational constant is approximately

$$G = 6.672 \times 10^{-11} \text{ N m}^2 \text{ kg}^{-2},$$

where the units are just to make everything work out.

The problem we want to consider is this: Suppose that the planet has radius R, and that object starts on the planet surface. We want to know how much energy it will take to lift the object from the planet's surface, and keep lifting it away "to ∞".



In physics, energy (in this case, in the form of "work") is calculated by integrating force over distance. In other words, we want to solve

$$\int_R^\infty \frac{GMm}{x^2}\, dx$$

The symbols G, M and m are just some numbers, so they aren't going to effect the integration, and we can just carry them along for the ride. So, what we're just worried about integrating $\int_R^\infty \frac{1}{x^2}\, dx$, and then multiplying the result by GMm.

2. The improper integral.

This is exactly the kind of integral we now know how to solve, using the ideas from last class. To understand an integral out "to ∞" we just understand it as a limit:

$$\lim_{b\to\infty} \int_R^b \frac{GMm}{x^2}\, dx$$

Since the antiderivative of $1/x^2$ is $-1/x$, we have

$$\left[-\frac{GMm}{x}\right]_{x=R}^{x=b} = GMm\left(\frac{1}{R} - \frac{1}{b}\right).$$

Letting $b \to \infty$ in the above equation gives us our answer:

$$\int_R^\infty \frac{GMm}{x^2}\, dx = \frac{GMm}{R}.$$

3. Using the answer.

The formula above gives the amount of energy needed to lift an object up from the planet's surface to a point infinitely far away. There are several fun ways to use this formula.

One way is this: Suppose we decide that we're going to throw the object very quickly, and give it a lot of energy all at once. How hard will we have to throw it in order that the object leaves the planet and never comes back?

This requires one more fact from physics: if an object of mass m is moving at velocity v, the energy (kinetic this time) of that object is $\frac{1}{2}mv^2$.

How large will v have to be so that $\frac{1}{2}mv^2$ is as much energy as $\frac{GMm}{R}$? That's easy:

$$\frac{1}{2}mv^2 = \frac{GMm}{R},$$

or

$$v = \sqrt{\frac{2GM}{R}}.$$

One nice thing about this formula is that this magic velocity doesn't depend on the mass of the object anymore, just the mass and radius of the planet.

This velocity is called the *escape velocity* of the planet. It's how fast something has to be going so that it will leave the planet and never come back.

For example, here are the escape velocities (as well as masses and radii) of some planets in the solar system.

Planet	M	R	Escape velocity
Mercury	3.30×10^{23} kg	2.440×10^6 m	$4,248$ m/s
Venus	4.869×10^{24} kg	6.051×10^6 m	$10,362$ m/s
Earth	5.972×10^{24} kg	6.378×10^6 m	$11,178$ m/s
Mars	6.422×10^{23} kg	3.396×10^6 m	$5,023$ m/s

So, for example, here on earth, if you throw something at a speed of more than 11.178 km/s, it's not coming back[1].

This handout can (soon) be found at

http://www.mast.queensu.ca/∼mikeroth/calculus/calculus.html

E-mail address: mikeroth@mast.queensu.ca

[1]what happens if you throw a boomerang that fast is a scientific mystery.