Integration by Parts

1. What we're up to.

We want to learn some techniques for integration. Our two main methods are the method of substitution, and the method of integration by parts.

The purpose of each method is to make the integral we're trying to solve simpler. We hope to use the method to transform our problem of integration into a problem we can solve right away, or failing that, a problem which is easier than the original one.

This week we are concentrating on integration by parts.

2. What it is.

Integration by parts is a pattern we can apply anytime the function we are trying to integrate can be written as a product.

There may be many ways that we can write the function as a product, and not all ways may be useful for simplifying the integral; some may even make the integral harder. Learning how to choose the "parts" is a matter of practice, and will become easier after trying a few problems.

Without worrying right now about how to choose the parts, let's first concentrate on understanding what the pattern is.

If we use "moving upwards" to represent integration, and "moving downwards" to represent differentiation, one way to represent the pattern is

$$\int f(x)g(x)\ dx = \overset{F(x)}{{\color{gray}f(x)}}g(x) - \int \overset{F(x)}{{\color{gray}f(x)}}\,\underset{g'(x)}{{\color{gray}g(x)}}\ dx.$$

The grey functions are just to remind us what used to be there, and aren't actually involved in the integral. We also don't really need to write the functions way up in the air, or below the center line, it's just a visual trick to try and make remembering the pattern easier.

As an example, here's integration by parts applied to $x\cos(x)$ using this visual notation:

$$\int \cos(x)\cdot x\ dx = \overset{\sin(x)}{{\color{gray}\cos(x)}}\cdot x - \int \overset{\sin(x)}{{\color{gray}\cos(x)}}\cdot \underset{1}{{\color{gray}x}}\ dx,$$

or, writing it in the ordinary way:

$$\int \cos(x)\cdot x\ dx = \sin(x)\cdot x - \int \sin(x)\cdot 1\ dx,$$

which is something we know how to solve.

There are many other ways to remember what the pattern is. A popular one is the "$u\,dv$" method:

$$\int u\,dv = uv - \int v\,du.$$

Here, the symbols mean

u – the piece which we will differentiate later.

dv – the piece which we'll integrate.

v – the integral of dv.

du – the derivative of u.

In the example of $\cos(x) \cdot x$, we picked:

$u = x$, and

$dv = \cos(x)$

from which we compute

$du = 1$, and

$v = \sin(x)$.

In the "$u\,dv$"pattern, this gives us

$$\int \cos(x) \cdot x \; dx = \sin(x) \cdot x - \int \sin(x) \cdot 1 \; dx,$$

just like before.

You can use any method you want to remember the pattern. Any way that works is a good way.

3. Levels of understanding.

Here are three levels of understanding of integration by parts:

Level 1: You can remember the integration by parts formula, and know how to apply it to solve all the examples we've done in class.

Level 2: You not only remember the formula, you understand why it's true, that is, you know why the left hand side of the equation is equal to the right hand side of the equation.

(There a bunch of ways to see this. One is the "inverse of product rule" argument in the course notes. Another is the "guess and fix" description from last class. You could also try and make up your own explanation.)

Level 3: You not only understand why the formula is true, you understand it well enough to explain it to somebody else.

Somewhere between level 2 and level 3 is a good goal to shoot for.

4. Picking the parts.

Figuring out how to pick the parts is a perpetual problem.

What we want is to choose the parts so that the integral we end up with is simpler than the one we started with. The only real strategy is a sort of advanced trial and error: Try and imagine what the end integral would be if you chose the pieces in various ways, then choose the way with the easiest answer.

Some experience using integration by parts will also make this easier.

This handout can (soon) be found at

http://www.mast.queensu.ca/∼mikeroth/calculus/calculus.html

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