Math 314 — Final Exam — April 2004

Instructions: The exam contains six questions, and you should hand in answers to all six of them.

You may freely use any notes or statements from class. It is also permissible to use outside sources, or to talk with other students about the exam questions, but if you do this be sure to acknowledge these discussions or references in your write up of those questions. You should, of course, try and do as much on your own as possible.

It is apparently mandatory to include the following statement:

> Please Note: The candidate is urged to submit with the answer paper a clear statement of any assumptions made if doubt exists as to the interpretation of any question that requires a written answer.

Another possible solution is just to ask me about the question.

The exam is due on or before **Friday, April 23, 2003**, at **4pm**.

1. [Medium answer] Suppose that $K \subseteq L$ is an extension of fields of finite degree.

(a) Explain the inductive argument showing that $|\mathrm{Aut}(L/K)| \leq [L : K]$. In the case that $|\mathrm{Aut}(L/K)| = [L : K]$ explain how the inductive step shows that $K \subseteq L$ is a separable normal extension.

(b) Conversely, assume that $K \subseteq L$ is a separable normal extension. Show how the inductive argument can be used to prove that $|\mathrm{Aut}(L/K)| = [L : K]$. You may use the lifting lemmas for normal extensions without proof, but be sure to mention how you're using them.

2. [Very short answer] Suppose that K is a field of characteristic zero (just so that we don't have to worry about separability issues), and that $p(x) \in K[x]$ is an irreducible polynomial over K, with splitting field L.

(a) If $\deg(p(x)) = 3$ what are the possibilities for the Galois group $G = \mathrm{Aut}(L/K)$? How can you distinguish between them?

(b) If $\deg(p(x)) = 4$ what are the possibilities for the Galois group $G = \mathrm{Aut}(L/K)$? How do you decide between them?

3. [Long answer] Let $\omega = e^{2\pi i/3}$ and $K = \mathbb{Q}(\omega)$. Let $p(x) \in K[x]$ be the polynomial $p(x) = x^4 + 0x^3 - 6x^2 + 9x - 3$; $p(x)$ is irreducible over K, a fact that you may assume without proof. Let L be the splitting field of p over K. Find $G = \mathrm{Aut}(L/K)$, and describe all the intermediate fields M, $K \subseteq M \subseteq L$ along with their inclusions.

This problem will likely take some organization, so please put some thought into what your notation will be, and in laying out your argument in a clean way.

One thing to note is that, if r is a positive real number, then the symbol $\sqrt{r}$ (or even $\sqrt[n]{r}$) has unambiguous meaning: it is the unique positive real number which is the square (or n-th) root of r. If c is a complex number, then the meaning of $\sqrt{c}$ is not so clear: there are two possibilities and no canonical way to distinguish between them. If you use a symbol like $\sqrt{c}$, be sure and indicate (perhaps by drawing a picture of the complex plane, or by specifying the "angle" of the root) which square root you mean. Any further time you use the symbol $\sqrt{c}$ I'll assume that you mean the same root, and that $-\sqrt{c}$ means the opposite root. Of course, if you have another complex number c' and want to write $\sqrt{c'}$, you'll have to go through the process all over again.

One last piece of advice: For any polynomial of degree n, we know a linear transformation of the variables which will get rid of the x^{n-1} term. This will likely be useful in finding the roots of the resolvent cubic.

4. [Short answer and follow up question]: Let $p(x)$, K, and G be as in question (3), and L' the splitting field of $p(x)$ over $\mathbb{Q}$. What is the relation of the Galois group G from question (3) and the Galois group $G' = \mathrm{Aut}(L'/\mathbb{Q})$? What is the field L'? What is the group G'? Only short answers are required here, but you should mention your justification for your statements.

5. [Very short answer.] If p is a prime number, show that the polynomial

$$x^{p-1} + x^{p-2} + \cdots + x + 1 = \frac{x^p - 1}{x - 1}$$

is irreducible over $\mathbb{Q}$. (Hint: Try the substitution $x = y + 1$.)

6. [Medium answer] Let $p(x) \in \mathbb{Q}[x]$ be the polynomial $x^{17} - 2$, and L the splitting field for $p(x)$ over $\mathbb{Q}$. Determine as much as you can about the Galois group $G = \mathrm{Aut}(L/\mathbb{Q})$. For instance, you should definitely find the order of G. If you can, give precisely the structure of the group G. If that seems impossible, consider just describing the structure of the p-Sylow subgroups of G. Can you describe all subgroups of G? How many are there of each order? (Note: You do *not*, and *should not* give all the intermediate fields corresponding to the subgroups of G.) Is the group G solvable? All claims should be justified, and the clarity of your explanation is also important.

Some Formulas

- A cubic $x^3 + bx^2 + cx + d$ has discriminant $b^2c^2 + 18bcd - 4b^3d - 4c^3 - 27d^2$.
- A quartic $x^4 + bx^3 + cx^2 + dx + e$ has resolvent cubic

$$t^3 - 2ct^2 + (c^2 + bd - 4e)t + (d^2 + b^2e - bcd).$$

- To compute the discriminant of a quartic, first compute its resolvent cubic, and then the discriminant of that cubic. This will be the same as the discriminant of the quartic computed directly.