Math 314 Homework Assignment 1 JANUARY 15, 2004

DUE DATE: JANUARY 29, 2004

1. Suppose that $F$ is a field and $B$ a ring, and $\phi : F \longrightarrow B$ a ring homomorphism. Show that $\phi$ is either an injection or the zero map. (If you use the convention that for ring homomorphisms, $1_F$ has to get sent to $1_B$, then this implies that all ring homomorphisms $\phi : F \longrightarrow B$ have to be injections, unless $B$ is the zero ring.)

Is the same thing true for ring homomorphisms $\phi : B \longrightarrow F$?

2. Suppose that $A$ is a ring, and $f_1$,..., $f_n$ any elements of $A$. Define the set

$$(f_1, \ldots, f_n) \stackrel{\text{def}}{=} \Big\{a_1 f_1 + a_2 f_2 + \cdots a_n f_n \;\Big|\; a_1, \ldots, a_n \in A\Big\} \subseteq A$$

to be the set of "linear combinations" of $f_1$, ..., $f_n$ with coefficients in $A$. i.e., the set of all possible sums of the above form.

Show that this is an ideal of $A$.

3. Suppose that $A$ is a ring with infinitely many elements with the following property. For every nonzero ideal $I$ of $A$, the quotient ring $A/I$ is *finite* (that is, it has only finitely many elements). Show that $A$ is a domain.

NOTE 1: Such rings exist: $A = \mathbb{Z}$ is an example.

NOTE 2: As always, we're assuming that our rings contain a 1. (The way that the "1" appears is slightly subtle and easy to overlook. So, if you have a solution, but it doesn't explicitly seem to use that fact that there's a 1 in the ring, don't worry about it, it doesn't necessarily mean that your solution is wrong).

NOTE 3: This problem is completely useless in practice. I have never seen a situation where you would apply the fact/lemma that you're supposed to prove above. It is very good for one thing though: you can't solve it without having a good picture of how quotient rings and ideals work.

4. Let $A$ be the ring $\mathbb{Q}[x]$, $f$ the polynomial $f = x^3 + 3x^2 + 3x - 1$, and $I$ the ideal of $A$ consisting of all multiples of $f$ (i.e. $I = (f)$ in the notation of question 2). Find representatives of degree $< 3$ in the ring $B = A/I$ for the following expressions:

(a) $x^4$.

(b) $(x^2 + 1)^3$.

(c) $1/x$.

5. Let $\alpha$ be the real number $\alpha = 2^{1/3} - 1$. To as many decimal places as you can (well, at least 6, and no more than 20), evaluate the following real numbers:

(a) $\alpha^4$.

(b) $(\alpha^2 + 1)^3$.

(c) $1/\alpha$.

(d) $\alpha^2 + 3\alpha + 3$.

(e) $24\alpha^2 + 60\alpha - 16$.

(f) $6\alpha^2 + 10\alpha - 3$.

6. A field $F$ with only finitely many elements is called a *finite field*. We know some (but not all) examples of these: for any prime number $p$, the field $\mathbb{F}_p$ has exactly $p$ elements.

Suppose that $F$ is a finite field. Explain why $\mathrm{char}(F)$ cannot be zero, and so must be some prime number $p$.

If $p = \mathrm{char}(F)$, show that the number of elements in $F$ is a power of $p$. This shows, for example, that there are no finite fields with 6 elements. Do you think it is possible to have a finite field with 4 elements?