Math 314 Homework Assignment 2 February 9, 2004

due date: February 23, 2004

1. Decide whether or not the following polynomials are irreducible over $\mathbb{Q}$:

(a) $3x^2 - 7x - 5$.

(b) $6x^3 - 3x - 18$.

(c) $x^3 - 7x + 1$.

(d) $x^3 - 9x - 9$.

2. Find the minimal polynomials for the following elements over $\mathbb{Q}$:

(a) $\alpha = \frac{1+\sqrt{5}}{2}$

(b) $\alpha = e^{\pi i/6}$

(c) $\alpha = e^{2\pi i/5}$

(d) $\alpha = 2\sin(\pi/9)$

Don't forget (in each case) to include some short argument stating why the polynomial you found is really irreducible over $\mathbb{Q}$.

3. Let L be the field $L = \mathbb{Q}(\omega)$ where $\omega = e^{2\pi i/3}$. Show that

$$L = \{a + b\omega \mid a, b \in \mathbb{Q}\},$$

and verify that the map $a + b\omega \mapsto (a - b) - b\omega$ is an automorphism of L.

4. Let α be the real number $\alpha = 2^{1/3}$, and $\beta = \alpha\omega$, where $\omega = e^{2\pi i/3}$ as above.

Are the fields $\mathbb{Q}(\alpha)$ and $\mathbb{Q}(\beta)$ the same subfields of $\mathbb{C}$? (that is, does $\mathbb{Q}(\alpha) = \mathbb{Q}(\beta)$?) Are they isomorphic as fields?

5. Is $L = \mathbb{Q}(\sqrt{2}, \sqrt{3})$ a simple extension? Give either a proof that it isn't, or an explicit element α in L and a proof that $\mathbb{Q}(\alpha) = L$.

6. An impossible problem.

In order to enter Moscow State University as an undergraduate, candidates were required to pass an oral examination by professors in their area of study (as well as other examinations in Russian, and political knowledge). For instance, to enter the mathematics department, the candidate would have to answer mathematical questions posed by some of the faculty.

Sometimes the examiners wanted a particular candidate to fail, and a special supply of difficult or cruel questions was kept in order to make this easier.

The following is one such question. It was stated as follows:

Question: Find rational numbers a, b, c, and d to solve

$$(a + b\sqrt{2}\,)^2 + (c + d\sqrt{2}\,)^2 = 2 + 2\sqrt{2}.$$

You can try expanding the squares and collecting terms, but it isn't so easy to see what to do (a, b, c, and d are in $\mathbb{Q}$, and not just integers).

Prove that this problem is impossible in the following simple way: Assume that there is a solution, apply a Galois automorphism to the equation, and use a single property of the real numbers to arrive at a contradiction.