1. Suppose that K is a field of characteristic zero, and $p(x) \in K[x]$ an irreducible polynomial of degree d over K. Let α_1, α_2, $\ldots$, α_d be the roots of $p(x)$, and $L = K(\alpha_1, \ldots, \alpha_d)$ the field obtained by joining all the roots of $p(x)$.

Let S be the set $S = \{\alpha_1, \ldots, \alpha_n\}$ of the roots.

(a) If σ is an element of $\mathrm{Aut}(L/K)$ explain why, for any root $\alpha_i \in S$, $\sigma(\alpha_i) \in S$ too, so that the group $G = \mathrm{Aut}(L/K)$ acts on the set S.

(b) If $\sigma \in G$, and $\sigma(\alpha_i) = \alpha_i$ for $i = 1, \ldots, d$, explain why σ is actually the identity map $\sigma : L \longrightarrow L$ on L.

(c) An action of a group G on a set S is the same as a homomorphism $G \longrightarrow \mathrm{Perm}(S)$ from G to the group of permutations of S. Explain why the action from part (a) gives an *injective* homomorphism.

(d) Explain why the group G acts *transitively* on S.

(e) Explain why the group G above can be realized as a subgroup of S_d, the symmetric group on d elements, such that the subgroup acts transitively on the set $\{1, \ldots, d\}$.

2. Let $K = \mathbb{Q}$, and $\alpha = e^{2\pi i/5}$. The polynomial $p(x) = x^4 + x^3 + x^2 + x + 1$ is the minimal polynomial for α over $\mathbb{Q}$.

(a) Show that $L = \mathbb{Q}(\alpha)$ is a/the splitting field for $p(x)$.

(b) If $\sigma \in \mathrm{Aut}(L/\mathbb{Q})$, show that σ is completely determined by what it does to α. (i.e., once you know what $\sigma(\alpha)$ is, you know how σ acts on all of L.)

(c) Compute the Galois group $G = \mathrm{Aut}(L/\mathbb{Q})$. (Keeping in mind part (d) of question 1 may help, but don't get hung up on it if it doesn't.)

(d) Describe the subgroups of G, and draw the corresponding diagram of intermediate fields between $\mathbb{Q}$ and L.

3. Let $K = \mathbb{Q}$ and $p(x) = x^3 - 2$. We have checked before (there are several ways) that $p(x)$ is irreducible over $\mathbb{Q}$. Let L be the splitting field for $p(x)$ over $\mathbb{Q}$.

Describe the field L, and find the Galois group $G = \mathrm{Aut}(L/K)$. Find all subgroups of G and draw the diagrams of the intermediate fields and subgroups. Include the indices of the field extensions, and the corresponding index of each subgroup in another in the diagrams. Finally, say which intermediate fields M are Galois extensions over $\mathbb{Q}$, and give their Galois groups $\mathrm{Aut}(M/\mathbb{Q})$.