MATH 110 Tutorial 2 Solutions

1. (a) $\left[\begin{array}{cc|c} 5 & 2 & 0 \\ 2 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 10 & 4 & 0 \\ 10 & 5 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 5 & 2 & 0 \\ 0 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right].$

   The rank is 2.

   (b) $\left[\begin{array}{rrr|r} 1 & 8 & -3 & 3 \\ -1 & 1 & 1 & 1 \\ 4 & 4 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{-29}{61} \\ 0 & 1 & 0 & \frac{28}{61} \\ 0 & 0 & 1 & \frac{4}{61} \end{array}\right].$

   The rank is 3.

   (c) $\left[\begin{array}{rrrr} 0 & -1 & -3 & 12 \\ -1 & 3 & 1 & 0 \\ 2 & -3 & 8 & 1 \end{array}\right] \sim \left[\begin{array}{rrrr} 1 & 0 & 0 & -332 \\ 0 & 1 & 0 & -123 \\ 0 & 0 & 1 & 37 \end{array}\right].$

   The rank is 3.

2. (a) has a unique solution, (b) has a unique solution, and (c) has infinitely many solutions corresponding to the free variable (the fourth).

3. In each case, we augment the vector $\vec{b}$ and determine the solutions to the corresponding system of equations.

   (a) $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is $0\vec{v}_1 + 0\vec{v}_2$.

   (b) $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is $\frac{1}{5}\vec{v}_1 + \frac{1}{5}\vec{v}_2$.

   (c) Inconsistent.

4. Consider the system of equations Is $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ a linear combination of

the vectors $\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ -4 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}$?

The corresponding matrix is $\begin{bmatrix} 1 & 3 & -1 & 1 \\ 3 & -4 & 1 & 0 \\ 5 & 0 & -3 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & \frac{13}{34} \\ 0 & 1 & 0 & \frac{9}{17} \\ 0 & 0 & 1 & \frac{33}{34} \end{bmatrix}$.

Therefore yes, it is a linear combination of the given vectors, with the weights given in the augment of the RREF matrix.

5. (*Challenge*) Let $S = \{u_1, \ldots, u_{n+1}\}$ with $u_j \in \mathbb{R}^n$. The set S is linearly independent if and only if there is only the trivial (all zero) solution to

$$c_1 u_1 + \ldots c_{n+1} u_{n+1} = \vec{0}.$$

(Why is this true?) The corresponding matrix for this equation is of dimension $n \times n + 2$ with the augmented column of zeros. We can get this by naming the entries of the column vectors u_j and expanding the equation of linear independence given above. Think of the c_j as the variables of the system. This system has more variables than equations, and hence it cannot have a unique solution.