MATH 110 Tutorial 3

- A transformation $T : \mathbb{R}^m \to \mathbb{R}^n$ is *linear* if for every $\vec{v}, \vec{w} \in \mathbb{R}^m$ and scalar k, it satisfies:

  1. $T(\vec{v} + \vec{w}) = T(\vec{v} + T(\vec{w})$, and

  2. $T(k\vec{v}) = kT(\vec{v})$.

- A linear transformation $T : \mathbb{R}^m \to \mathbb{R}^n$ can be expressed by the map $T(\vec{x}) = A\vec{x}$ for some $n \times m$ matrix A.

- In $\mathbb{R}^2$, rotations and reflections are examples of linear transformations from the space to itself. The matrix representing a **counter-clockwise** rotation of θ is given by $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.

- The *inverse* of a matrix A (usually denoted A^{-1}) gives the reverse map of A so that applying A and then A^{-1} results in the identity. The inverse of A does not always exist.

1. Suppose $T_1, T_2, T_3, T_4 : \mathbb{R}^2 \to \mathbb{R}^3$ yield the following images.

(x,y)	$T_1(x,y)$
$[0,0]$	$[1,0,0]$
$[11,0]$	$[1,2,-1]$
$[0,-6]$	$[-1,1,4]$

(x,y)	$T_2(x,y)$
$[1,0]$	$[1,0,0]$
$[0,1]$	$[1,1,1]$
$[2,2]$	$[3,2,2]$

(x,y)	$T_3(x,y)$
$[1,2]$	$[3,5,9]$
$[-1,2]$	$[1,0,1]$
$[1,1]$	$[1,1,1]$

(x,y)	$T_4(x,y)$
$[5,2]$	$[1,2,3]$
$[1,2]$	$[0,1,2]$
$[3,-1]$	$[1,1,1]$

Is it possible that these maps are linear? If so, give the matrices that represent the linear transformations.

2. Let $T : \mathbb{R}^4 \to \mathbb{R}^2$ be the function which sends a polynomial (of degree 3) to its second derivative. Show that T is a linear transformation.

3. Let $f(x,y) := x^2 + y^2$, and $g(x,y) := xy$. Define $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$T(x,y) = \begin{bmatrix} f(x+2, y+3) - f(x,y) - f(2,3) \\ g(x-1, y+5) - g(x,y) - g(-1,5) \end{bmatrix}.$$

Determine whether T is a linear transformation or not.

4. In $\mathbb{R}^2$ draw the points with integer coordinates $-2 \leq x, y \leq 2$. There are 25 of them. Label these points $a, b, \ldots$. In another copy of $\mathbb{R}^2$, determine the image of the previous set of dots under the linear transformation induced by the matrix $\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$. Describe in words the affect of this linear map.

5. *(Challenge).* Let $T : \mathbb{R}^m \to \mathbb{R}^n$ be a linear transformation. The collection of points in $\mathbb{R}^m$ whose image under T is $\vec{0}$ is called the *kernel* of T, and is denoted $\ker(T)$. Give an example of a linear transformation whose kernel is the plane $x+2y+3z = 0$ in $\mathbb{R}^3$. In general, suppose you are given a kernel as a linear equation as above. Find a linear transformation with this kernel when it is possible, and determine when, if ever, it is impossible.