MATH 110 Tutorial 3 Solutions

1. (a) No; In any linear transformation $\vec{0} \mapsto \vec{0}$, since by linearity $T(0 \cdot \vec{0}) = 0 \cdot T(\vec{0} = \vec{0}$.

   (b) Yes, since $T([2,2]) = T(2 \cdot [1,0] + 2 \cdot [0,1])$, T can be extended to a linear map.

   (c) No; note that $\begin{bmatrix} -1 \\ 2 \end{bmatrix} = -4 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Hence for linearity,
$$T\left(\begin{bmatrix} -1 \\ 2 \end{bmatrix}\right) = T\left(-4 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} 1 \\ 2 \end{bmatrix}\right).$$
   However, RHS $= \begin{bmatrix} -5 \\ -11 \\ 14 \end{bmatrix} \neq$ LHS $= \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

   (d) No; if T_4 was linear, then we can deduce from the first pair that $(1,0) \mapsto (\frac{1}{4}, \frac{1}{4}, \frac{1}{4})$. From the last pair we can deduce that $(1,0) \mapsto (\frac{2}{7}, \frac{3}{7}, \frac{4}{7})$, a contradiction.

2. There is nothing particular about the numbers $4, 2$. It is true in general that if k is a constant, that
$$\frac{d}{dt}(k \cdot f(t)) = k \frac{d}{dt} f(t),$$
and
$$\frac{d}{dt}(f(t) + g(t)) = \frac{d}{dt} f(t) + \frac{d}{dt} g(t).$$
These are the two properties required for linearity.

3. Although it may not appear at first glance, T is indeed linear:
$$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} (x+2)^2 + (y+3)^2 - x^2 - y^2 - 13 \\ (x-1)^2 + (y+5)^2 - x^2 - y^2 - 26 \end{bmatrix} = \begin{bmatrix} 4x + 6y \\ -2x + 10y \end{bmatrix}.$$
This is clearly a linear map, and T can be expressed as $T(x) = Ax$ for
$$A := \begin{bmatrix} 4 & 6 \\ -2 & 10 \end{bmatrix}.$$

4. The resulting picture is called a *shear*. Sorry, I don't know how to draw the picture in LaTex.

5. *Challenge.* Hopefully this was an easy question! An example of a linear system whose kernel is the plane $x + 2y + 3z = 0$ is the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}$ given by $T(x, y, z) = x+2y+3z$. Clearly the kernel is the set of points on the plane $x+2y+3z = 0$. In general, if you are given a kernel equation $f(x_1, \ldots x_n) = 0$, the linear transformation which is defined by $T(x_1, \ldots x_n) = f(x_1, \ldots x_n)$ will satisfy the condition. Naturally you should ask here, "What if the kernel equation was $\ldots = 1$, instead of zero?" This can never be; I leave this as an exercise for you to verify.