MATH 110 Tutorial 4

- A function f is *invertible* if $f(x) = c$ has a unique solution x for every c.
- In different branches of mathematics when we are describing functions we use the words **invertible**, **one-to-one**, **injection**, and **monomorphism** interchangeably.
- An $n \times m$ matrix is invertible if and only if $m = n$ and $\text{rref}(A) = I_n$, that is, when $\text{rank}(A) = n$.
- Not all matrices are invertible!
- A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is invertible if and only if the only point mapped to $\vec{0}$ is $\vec{0}$. (The collection of points mapped to $\vec{0}$ is so important in algebra that we give it a special name, the *kernel*. We denote the kernel of T by $\ker(T) = \{x : T(x) = \vec{0}\}$. More on this in a later section ...)
- Matrix multiplication AB is defined only if A has the same number of columns as B has rows. The resulting matrix AB will have the same number of rows as A and the same number of columns as B. This means that if we write the dimensions of A, B side-by-side, we can only multiply if the middle two numbers are the same. In this case, the dimensions of AB are what is left when we cross out the middle matching numbers.
- Matrix multiplication:

  1. Associates: $(AB)C = A(BC)$.
  2. Distributes: $A(B + C) = AB + AC$.
  3. Does **NOT** commute: $AB \neq BA$ in general. *e.g*:

  $$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}.$$

  $$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}.$$

- $AA^{-1} = A^{-1}A = I_n$.

1. Compute the following matrix products.

   (a) $\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 0 & 3 \\ 1 & 1 \end{bmatrix}$, (b) $\begin{bmatrix} 1 & 3 & 1 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 3 & 1 \end{bmatrix}$, (c) $\begin{bmatrix} 1 & -2 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$.

2. Calculate the inverse of each of the following matrices. Multiply the matrix with its inverse to verify your calculation.

   (a) $\begin{bmatrix} 1 \end{bmatrix}$, (b) $\begin{bmatrix} 3 & -1 \\ -2 & -1 \end{bmatrix}$, (c) $\begin{bmatrix} 1 & 2 & 3 \\ -1 & -1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$, (d) $\begin{bmatrix} -1 & 2 & -1 \\ -1 & 3 & 0 \\ 1 & 1 & -4 \end{bmatrix}$.

3. *(Challenge)*. Deciding whether something has an inverse is a pivotal question in algebra. Determine a simple test for whether a 2×2 matrix is invertible; hopefully you will obtain what is called the *determinant* of the matrix. Suppose you are given a 2×2 matrix with integer coefficients. Under what condition(s) will the inverse matrix also have integer coefficients?