MATH 110 Tutorial 4

1. Compute the following matrix products.

   (a) $\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 0 & 3 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 3 & 6 \end{bmatrix}$,

   (b) $\begin{bmatrix} 1 & 3 & 1 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 2 \\ 3 & 2 \end{bmatrix}$,

   (c) $\begin{bmatrix} 1 & -2 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$.

2. Calculate the inverse of each of the following matrices. Multiply the matrix with its inverse to verify your calculation.

   (a) $\begin{bmatrix} 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 \end{bmatrix}$.

   (b) $\begin{bmatrix} 3 & -1 \\ -2 & -1 \end{bmatrix}$. Augment I_2 and reduce into rref.

   $$\left[\begin{array}{cc|cc} 3 & -1 & 1 & 0 \\ -2 & -1 & 0 & 1 \end{array}\right] = \left[\begin{array}{cc|cc} 6 & -2 & 2 & 0 \\ -6 & -3 & 0 & 3 \end{array}\right] = \left[\begin{array}{cc|cc} 3 & -1 & 1 & 0 \\ 0 & -5 & 2 & 3 \end{array}\right] =$$

   $$\left[\begin{array}{cc|cc} 15 & -5 & 5 & 0 \\ 0 & -5 & 2 & 3 \end{array}\right] = \left[\begin{array}{cc|cc} 15 & 0 & 3 & -3 \\ 0 & -5 & 2 & 3 \end{array}\right] = \left[\begin{array}{cc|cc} 1 & 0 & \frac{1}{5} & \frac{-1}{5} \\ 0 & 1 & \frac{-2}{5} & \frac{-3}{5} \end{array}\right].$$

   Hence the inverse is $\begin{bmatrix} \frac{1}{5} & \frac{-1}{5} \\ \frac{-2}{5} & \frac{-3}{5} \end{bmatrix}$.

   (c) $\begin{bmatrix} 1 & 2 & 3 \\ -1 & -1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$. Augment I_3 and reduce into rref.

   $$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 1 & 0 \\ 2 & 0 & 1 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 3 & 1 & 1 & 0 \\ 0 & -2 & 1 & 0 & 2 & 1 \end{array}\right] = \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 3 & 1 & 1 & 0 \\ 0 & -2 & 1 & 0 & 2 & 1 \end{array}\right] =$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 4 & 1 & 2 & 1 \\ 0 & 1 & 3 & 1 & 1 & 0 \\ 0 & 0 & 7 & 2 & 4 & 1 \end{array}\right] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{-1}{7} & \frac{-2}{7} & \frac{3}{7} \\ 0 & 1 & 0 & \frac{1}{7} & \frac{-5}{7} & \frac{-3}{7} \\ 0 & 0 & 0 & \frac{2}{7} & \frac{4}{7} & \frac{1}{7} \end{array}\right].$$

Hence the inverse is $\begin{bmatrix} \frac{-1}{7} & \frac{-2}{7} & \frac{3}{7} \\ \frac{1}{7} & \frac{-5}{7} & \frac{-3}{7} \\ \frac{2}{7} & \frac{4}{7} & \frac{1}{7} \end{bmatrix}$.

(d) $\begin{bmatrix} -1 & 2 & -1 \\ -1 & 3 & 0 \\ 1 & 1 & -4 \end{bmatrix}$. Augment I_2 and reduce into rref.

$$\left[\begin{array}{ccc|ccc} 1 & 1 & -4 & 0 & 0 & 1 \\ 0 & 4 & -4 & 0 & 1 & 1 \\ 0 & 3 & -5 & 1 & 0 & 1 \end{array}\right] = \left[\begin{array}{ccc|ccc} 1 & 1 & -4 & 0 & 0 & 1 \\ 0 & 12 & -12 & 0 & 3 & 3 \\ 0 & 12 & -20 & 4 & 0 & 4 \end{array}\right] = \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 3 & 1 & 1 & 0 \\ 0 & 0 & 8 & -4 & 3 & -1 \end{array}\right] =$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{-3}{2} & \frac{7}{8} & \frac{3}{8} \\ 0 & 1 & 0 & \frac{-1}{2} & \frac{5}{8} & \frac{1}{8} \\ 0 & 0 & 0 & \frac{-1}{2} & \frac{3}{8} & \frac{-1}{8} \end{array}\right].$$

Hence the inverse is $\begin{bmatrix} \frac{-3}{2} & \frac{7}{8} & \frac{3}{8} \\ \frac{-1}{2} & \frac{5}{8} & \frac{1}{8} \\ \frac{-1}{2} & \frac{3}{8} & \frac{-1}{8} \end{bmatrix}$.

3. *(Challenge)*. Deciding whether something has an inverse is a pivotal question in algebra. Determine a simple test for whether a 2×2 matrix is invertible; hopefully you will obtain what is called the *determinant* of the matrix. Suppose you are given a 2×2 matrix with integer coefficients. Under what condition(s) will the inverse matrix also have integer coefficients?

$$\left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right] = \left[\begin{array}{cc|cc} ac & bc & c & 0 \\ ac & ad & 0 & a \end{array}\right] = \left[\begin{array}{cc|cc} a & b & 1 & 0 \\ 0 & ad-bc & -c & a \end{array}\right] =$$

$$\left[\begin{array}{cc|cc} a & b & 1 & 0 \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right] = \left[\begin{array}{cc|cc} a & b & 1 & 0 \\ 0 & b & \frac{-bc}{ad-bc} & \frac{ab}{ad-bc} \end{array}\right] = \left[\begin{array}{cc|cc} a & 0 & \frac{ad}{ad-bc} & \frac{-ab}{ad-bc} \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right] =$$

$$\left[\begin{array}{cc|cc} 1 & 0 & \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right].$$

Hence the inverse is $\begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. This exists if and only if $ad - bc \neq 0$. Let $A := \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with a, b, c, d integers.

Claim. For the inverse of the matrix A to have integer coefficients it is **sufficient** and **necessary** that $ad - bc = \pm 1$.

Proof. The sufficiency is clear from the calculation of the inverse of A above. We will verify the necessity. Suppose that $ad - bc$ divides a, b, c, d. Then $(ad - bc)^2$ divides ad and bc, and hence $(ad - bc)^2$ divides $ad - bc$. This implies that $ad - bc$ divides 1, and hence $ad - bc = \pm 1$. $\square$

Several people were able to say in the tutorials that if $ad - bc = \pm 1$ then A^{-1} has integer entries, but no one was able to prove the converse. I hope my proof is clear.