MATH 110 Tutorial 5

- The *image* of a function is the collection of all points hit in the codomain.
- The image of a linear transformation T, denoted $\text{im}(T)$, is closed under scalar multiplication and vector addition, and $\vec{0} \in \text{im}(T)$.
- The image of a linear transformation is a subspace of the codomain.
- The *span* of a collection S of vectors is the set of all linear combinations of vectors of S. If $S = \{\vec{x}_1, \ldots, \vec{x}_k\}$ then

$$\text{span}(S) = \left\{ \sum_{n=1}^{k} c_n \vec{x}_n : c_n \in \mathbb{R} \right\}.$$

- The *kernel* of a linear transformation T is the set of elements mapped to $\vec{0}$ under the linear transformation. We write

$$\ker(T) = \{x : T(x) = \vec{0}\}.$$

- For a linear transformation T, $\ker(T)$ is closed under scalar multiplication and vector addition, and $\vec{0} \in \ker(T)$. When the kernel consists solely of the zero-vector, we call the kernel *trivial*.
- For a linear transformation T represented by the $n \times n$ matrix A, the following are equivalent.
  - A is invertible.
  - $A\vec{x} = \vec{b}$ has a unique solution $\vec{x} \in \mathbb{R}^n$ for each $\vec{b} \in \mathbb{R}^n$.
  - $\text{rref}(A) = I_n$.
  - $\text{rank}(A) = n$.
  - $\text{im}(T) = \mathbb{R}^n$.
  - $\ker(T) = \{\vec{0}\}$.

1. For each of the following matrices, compute the kernel of the corresponding linear transformation.

(a) $\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$, (b) $\begin{bmatrix} 1 & 3 \\ 1 & 3 \end{bmatrix}$, (c) $\begin{bmatrix} 1 & 3 & 1 \\ 2 & 1 & 0 \end{bmatrix}$, (d) $\begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 3 & 1 \end{bmatrix}$.

(e) $\begin{bmatrix} 1 & -2 & 1 & 3 \end{bmatrix}$, (f) $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$.

2. Each of the following matrices represents a linear transformation from $\mathbb{R}^2$ to itself. Compute the images and kernels, and interpret them geometrically. Is there a connection between image and kernel?

(a) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, (b) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, (c) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, (d) $\begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}$.

3. Let A, B be matrices, so that the product AB exists. If $\ker(A) = \ker(B) = \{0\}$, then find $\ker(AB)$.

4. *(Challenge)*. The linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is invertible if and only if $\ker(T) = \{\vec{0}\}$. Prove this fact. (Note that proving an equivalence requires you to prove two implications. Assume that T is invertible and then prove that its kernel is trivial, and then assume that T has trivial kernel, and prove that T is invertible. Please make sure that it is a **proof**, and not just a heuristic argument.)