MATH 110 Tutorial 5

1. For each of the following matrices, compute the kernel of the corresponding linear transformation. We will solve for the kernel and image simultaneously.

   (a) $\left[\begin{array}{cc|cc} 2 & -1 & u & 0 \\ 1 & 3 & v & 0 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & \frac{3a-b}{7} & 0 \\ 0 & 1 & \frac{2b-a}{7} & 0 \end{array}\right]$. The kernel equations are $x = 0, y = 0$, and hence is trivial. The system is consistent, and hence image is the entire space of $\mathbb{R}^2$.

   (b) $\left[\begin{array}{cc|cc} 1 & 3 & u & 0 \\ 1 & 3 & v & 0 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 3 & u & 0 \\ 0 & 0 & u-v & 0 \end{array}\right]$. The kernel equations are $x + 3y = 0$ and $0 = 0$. Hence the kernel is the span of $\begin{bmatrix} -3 \\ 1 \end{bmatrix}$. The system is consistent if and only if $u - v = 0$, and hence the image corresponds to the line $x = y$.

   (c) $\left[\begin{array}{ccc|cc} 1 & 3 & 1 & u & 0 \\ 2 & 1 & 0 & v & 0 \end{array}\right] \sim \left[\begin{array}{ccc|cc} 1 & 0 & -\frac{1}{5} & \frac{3v-u}{5} & 0 \\ 0 & 1 & \frac{2}{5} & \frac{2u-v}{5} & 0 \end{array}\right]$. The kernel equations are $x - \frac{1}{5}z = 0, y + \frac{2}{5}z = 0$, and hence the kernel is the span of $\begin{bmatrix} 5 \\ -\frac{5}{2} \\ 1 \end{bmatrix}$. The system is consistent, and so the image is all of $\mathbb{R}^2$.

   (d) $\left[\begin{array}{cc|cc} 1 & 1 & u & 0 \\ 1 & 0 & v & 0 \\ 3 & 1 & w & 0 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & v & 0 \\ 0 & 1 & u-v & 0 \\ 0 & 0 & w-u-2v & 0 \end{array}\right]$. The kernel equations are $x = 0, y = 0, 0 = 0$. Hence the kernel is trivial. The system is consistent if and only if $w - u - 2v = 0$, and hence the image is the span of $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \right\}$; a plane.

(e) $\left[\begin{array}{cccc|cc} 1 & -2 & 1 & 3 & u & 0 \end{array}\right]$ is already in RREF. The kernel equation is $x-2y+z+3w=0$ which corresponds to the span of $\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

The system is consistent and so the image is $\mathbb{R}$.

(f) $\left[\begin{array}{c|cc} 1 & s & 0 \\ 1 & t & 0 \\ 1 & u & 0 \\ 0 & v & 0 \end{array}\right] \sim \left[\begin{array}{c|cc} 1 & s & 0 \\ 0 & s-t & 0 \\ 0 & s-u & 0 \\ 0 & v & 0 \end{array}\right]$. The kernel equations are $x=0, 0=0, 0=0, 0=0$. Hence the kernel is the $y-axis$. The system is consistent if and only if $s-t=s-u=v=0$, or equivalently the only points hit are those of the form $(s,s,s,0)$ for real s.

2. Each of the following matrices represents a linear transformation from $\mathbb{R}^2$ to itself. Compute the images and kernels, and interpret them geometrically. Is there a connection between image and kernel?

(a) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. The kernel equations are $x = 0, y = 0$. Any augmented system is consistent. Hence the image is all of $\mathbb{R}$, and the kernel is trivial; the origin.

(b) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. The kernel equations are $x = 0, 0 = 0$ and so the kernel is the y-axis. The system is consistent only if the second coordinate in the image is zero; hence the image is the collection of points $(u, 0)$ in $\mathbb{R}^2$, the x-axis.

(c) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. The kernel equations are $0 = 0, 0 = 0$, and hence everything in $\mathbb{R}^2$ is in the kernel. This makes sense, because clearly if we multiply anything by this matrix the result will be $\vec{0}$; hence this is the image. We can verify this because an augmented system will be consistent only if both coordinates are zero.

(d) $\left[\begin{array}{cc|cc} 2 & 1 & u & 0 \\ 2 & 1 & v & 0 \end{array}\right] \sim \left[\begin{array}{cc|cc} 2 & 1 & u & 0 \\ 0 & 0 & u - v & 0 \end{array}\right]$. This system is consistent if and only if $u = v$, and so the image is the collection of points (u, u) for u real. The kernel equations are $2x + y = 0, 0 = 0$, and hence the kernel is exactly this line, the span of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

3. Let A, B be matrices, so that the product AB exists. If $\ker(A) = \ker(B) = \{0\}$, then find $\ker(AB)$.

   Recall that matrix multiplication associates.

$$\begin{aligned} \ker(AB) &= \{x : (AB)x = \vec{0}\} \\ &= \{x : A(Bx) = \vec{0}\} \\ &= \{y : Ay = \vec{0}, Bx = y\}. \end{aligned}$$

   Now we use the fact that the kernel of A is trivial; this means that $Ay = \vec{0}$ implies that $y = \vec{0}$, so that

$$\ker(AB) = \{x : Bx = \vec{0}\}.$$

   However we also know that the kernel of B is trivial, and hence this set contains only the element $\vec{0}$, as required.

4. *(Challenge)*. The linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is invertible if and only if $\ker(T) = \{\vec{0}\}$. Prove this fact. (Note that proving an equivalence requires you to prove two implications. Assume that T is invertible and then prove that its kernel is trivial, and then assume that T has trivial kernel, and prove that T is invertible. Please make sure that it is a **proof**, and not just a heuristic argument.)

   $\Rightarrow$: Suppose T is invertible. Then each point has a unique image, since the inverse of T is well-defined; $T^{-1}(\vec{0})$ is a single point because it is a function. Alternatively, the matrix for T must have a pivot in every column, and hence the kernel equations leave no free variable.
   $Leftarrow$: Suppose $\ker(T) = \{\vec{0}\}$. A square matrix is invertible if and only if it has a pivot in every column. If the kernel is trivial, then the kernel equations leave no free variable, and hence every variable is a pivot. This implies that T is invertible.