1.

(a) Write $\vec{b}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\vec{b}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and $\vec{b}_3 = \begin{bmatrix} 11 \\ 2 \end{bmatrix}$ as linear combinations of $\begin{bmatrix} 4 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

(b) Suppose that T is a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^3$, and that

$$T\left(\begin{bmatrix} 4 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}, \text{ and } T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}.$$

Find

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right), \ T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right), \text{ and } T\left(\begin{bmatrix} 11 \\ 2 \end{bmatrix}\right),$$

and explain how you know that T has to do this.

(c) Find the matrix A representing the linear transformation T.

(d) Compute the vectors $A\vec{b}_1$, $A\vec{b}_2$, and $A\vec{b}_3$.

2. The cross product of two vectors (a_1, a_2, a_3) and (b_1, b_2, b_3) in $\mathbb{R}^3$ is defined by

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \times \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{bmatrix}.$$

Consider an arbitary vector $\vec{v} = (v_1, v_2, v_3)$ in $\mathbb{R}^3$. Is the transformation $T(\vec{x}) = \vec{v} \times \vec{x}$ linear? If so, compute its matrix A in terms of the components of $\vec{v}$.

3. I'd like to consider some linear transformations from $\mathbb{R}^3$ to $\mathbb{R}^3$.

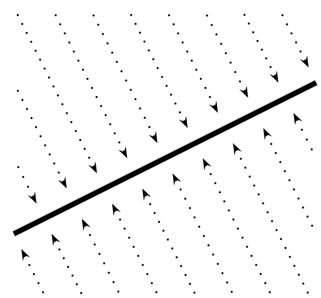
(a) Write down the matrix for the linear transformation T_1 which rotates by $\pi/4$ clockwise around the z-axis. (Clockwise means that, if we're at a point on the z-axis where the z values are positive, looking down at the xy-plane, the vectors in the xy-plane are rotated clockwise).

(b) Write down the matrix for the linear transformation T_2 which rotates by $\pi/2$ counterclockwise around the x-axis (again, this means that looking at the yz-plane from a point on the x-axis where the x values are positive, the vectors in the yz-plane are rotated counterclockwise).

(c) Write down the matrix for the linear transformation T_3 which rotates by $\pi/4$ clockwise around the y-axis. (Similar interpretation as above.)

(d) If I start with vector $\vec{e}_1 = (1, 0, 0)$, and apply transformation T_1, then T_2 to the answer, and then finally apply T_3, what vector in $\mathbb{R}^3$ will I end up with? Compute what will happen if I also do this to the vectors $\vec{e}_2 = (0, 1, 0)$ and $\vec{e}_3 = (0, 0, 1)$.

4. Given a line L in $\mathbb{R}^2$, *projection onto* L is the function $\mathbb{R}^2 \longrightarrow \mathbb{R}^2$ which sends every point in $\mathbb{R}^2$ to the nearest point on L, as shown in the diagram at right.



For any m, let $T_m : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the projection onto the line with slope m through the origin. This turns out to be a linear map, something you can assume when doing the question.

(a) Find the matrix for T_m, and explain your steps.

(b) What is the rank of this matrix?

(c) Let's define a function $T'_m : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ by

$$T'_m(\vec{v}) = \vec{v} - T_m(\vec{v})$$

for any $\vec{v} \in \mathbb{R}^2$.

Show that this is also a linear function from $\mathbb{R}^2$ to $\mathbb{R}^2$.

(d) Explain geometrically what T'_m does.

(e) Find the matrix for T'_m. (Do this in the usual way, by seeing what happens to $\vec{e}_1$ and $\vec{e}_2$.)

(f) The answers for (d) and (a) give another way to compute T'_m; use this to check your computations in (e).

(g) As $m \to \infty$, what happens to the line of slope m? What happens to the matrix associated to T_m? Does this make sense?