1. The RREF's are:

(a) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}$, (b) $\begin{bmatrix} 0 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 3/2 \end{bmatrix}$,

(c) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix}$, (d) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 6 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

2. If $f(t) = a + bt + ct^2 + dt^3$, then the condition that $f(0) = 1$ is the same as $a+b\cdot 0+c\cdot 0^2+d\cdot 0^3 = 1$, or $a = 1$. The condition that $f(1) = 0$ is $a+b\cdot 1+c\cdot 1^2+d\cdot 1^3 = 0$ or $a + b + c + d = 0$. Similarly, the other two conditions also give linear equations in a, b, c, and d. The system to be solved is:

$$\begin{aligned} a &= 1 \\ a + b + c + d &= 0 \\ a - b + c - d &= 0 \\ a + 2b + 4c + 8d &= -15 \end{aligned}$$

Representing this by a matrix, and putting the matrix into RREF, we get

$$\left[\begin{array}{rrrr|r} 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & -1 & 1 & -1 & 0 \\ 1 & 2 & 4 & 8 & -15 \end{array}\right] \rightsquigarrow \left[\begin{array}{rrrr|r} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -2 \end{array}\right],$$

giving the unique solution $a = 1$, $b = 2$, $c = -1$, and $d = -2$, or the polynomial

$$f(t) = 1 + 2t - t^2 - 2t^3.$$

It's a good habit to check that this actually goes through the points it's supposed to.

3. Writing down the matrix representing the equations

$$\left[\begin{array}{rrrrr|r} 1 & -7 & 0 & 0 & 1 & 3 \\ 0 & 0 & 1 & 0 & -2 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{array}\right]$$

we see that it is already in RREF. The leading (or dependent) variables are x_1, x_3, and x_4, while the free variables are x_2 and x_5. The general solution is given by

$$\begin{aligned} x_1 &= 3+7t_1-t_2 \\ x_2 &= t_1 \\ x_3 &= 2+2t_2 \\ x_4 &= 1-t_2 \\ x_5 &= t_2 \end{aligned} \qquad \text{or} \qquad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix} + t_1 \begin{bmatrix} 7 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} -1 \\ 0 \\ 2 \\ -1 \\ 1 \end{bmatrix}$$

4. *The Hundred Fowl problem.*

(a) The equations are

$$\begin{aligned} 5r+3h+1/3c &= 100 \\ r+h+c &= 100 \end{aligned}$$

(b) Representing these equations by a matrix, and putting the matrix into RREF, we get

$$\left[\begin{array}{ccc|c} 5 & 3 & 1/3 & 100 \\ 1 & 1 & 1 & 100 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & -4/3 & -100 \\ 0 & 1 & 7/3 & 200 \end{array}\right]$$

The leading (or dependent) variables are r and h, and the independent variable c. The general solution is given by

$$\begin{aligned} r &= -100+4/3t \\ h &= 200-7/3t \\ c &= t \end{aligned} \qquad \text{or} \qquad \begin{bmatrix} r \\ h \\ c \end{bmatrix} = \begin{bmatrix} -100 \\ 200 \\ 0 \end{bmatrix} + t \begin{bmatrix} 4/3 \\ -7/3 \\ 1 \end{bmatrix}.$$

(c) In order that r and h be integers, we need t to be a multiple of 3. In order that r be positive, we need $-100+4/3t > 0$ or $t > 75$. In order for h to be positive, we also need $200-7/3t > 0$, or $600/7 > t$.

What are the multiples of 3 greater than 75 and less than $600/7 \cong 85.71428$? The only possibilities are $t = 78$, 81, and 84.

This gives three solutions to the original problem.

When $t = 78$, we have $(r,h,c) = (4,18,78)$.

When $t = 81$, we have $(r,h,c) = (8,11,81)$.

When $t = 84$, we have $(r,h,c) = (12,4,84)$.

The analysis also shows that these are the only solutions to the problem.

5. Let's start by getting the matrix as close to RREF as we can without having to worry about exactly what k and ℓ are:

$$\left[\begin{array}{ccc|c} 1 & 1 & -2 & 1 \\ 2 & k & 1 & 2 \\ 1 & 10 & k & \ell \end{array}\right] \overset{\substack{R2-2\cdot R1 \\ R3-R1}}{\rightsquigarrow} \left[\begin{array}{ccc|c} 1 & 1 & -2 & 1 \\ 0 & k-2 & 5 & 0 \\ 0 & 9 & k+2 & \ell-1 \end{array}\right] \overset{\text{swap } R2 \text{ and } R3}{\rightsquigarrow}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -2 & 1 \\ 0 & 9 & k+2 & \ell-1 \\ 0 & k-2 & 5 & 0 \end{array}\right] \overset{\text{divide } R2 \text{ by } 9}{\rightsquigarrow} \left[\begin{array}{ccc|c} 1 & 1 & -2 & 1 \\ 0 & 1 & \frac{k+2}{9} & \frac{\ell-1}{9} \\ 0 & k-2 & 5 & 0 \end{array}\right] \overset{R3-(k-2)R2}{\rightsquigarrow}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -2 & 1 \\ 0 & 1 & \frac{k+2}{9} & \frac{\ell-1}{9} \\ 0 & 0 & \frac{49-k^2}{9} & \frac{(2-k)(\ell-1)}{9} \end{array}\right]$$

At this point, we can see that the critical entry for deciding whether or not the system will have any solutions is the entry $\frac{49-k^2}{9}$. If this is zero, and the entry $\frac{(2-k)(\ell-1)}{9}$ not zero, then there won't be any solutions. If $\frac{49-k^2}{9}$ is nonzero, then there will be a unique solution. Finally if $\frac{49-k^2}{9}$ is zero, and $\frac{(2-k)(\ell-1)}{9}$ is also zero, then there will be a one parameter family of solutions.

So, the answers to the questions are

(a) As long as $\frac{49-k^2}{9} \neq 0$, i.e., as long as $k \neq 7$ and $k \neq -7$, there is exactly one solution.

(b) If $k = 7$ or $k = -7$, and $\ell \neq 1$, then there are no solutions.

(c) If $k = 7$ and $\ell = 1$, then the matrix becomes $\left[\begin{array}{ccc|c} 1 & 1 & -2 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & -3 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$.

The general solution is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$$

If $k = -7$ and $\ell = 1$, then the matrix becomes $\left[\begin{array}{ccc|c} 1 & 1 & -2 & 1 \\ 0 & 1 & \frac{-5}{9} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & \frac{-13}{9} & 1 \\ 0 & 1 & \frac{-5}{9} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

With general solution

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} \frac{13}{9} \\ \frac{5}{9} \\ 1 \end{bmatrix}.$$