1. To deal with these problems about span and dimension, it might be helpful to use the following, which was proved in class:

Linear Lemma — If M is a $p \times q$ matrix with $q > p$, then there is a nonzero vector in the kernel of M.

(a) If V is a subspace of $\mathbb{R}^n$, and $w_1, \ldots, w_p$ are vectors which span V, then every vector v in V can be written as a linear combination of $w_1, \ldots, w_p$.

In particular since v_1 is in V, there are constants $c_{1,1}, \ldots, c_{p,1}$ with $v_1 = c_{1,1}w_1 + c_{2,1}w_2 + \cdots + c_{p,1}w_p$. Similarly, since v_2 is in V, there are constants $c_{1,2}, \ldots, c_{p,2}$ with $v_2 = c_{1,2}w_1 + c_{2,2}w_2 + \cdots + c_{p,2}w_p$, and in general constants $c_{1,i}, \ldots, c_{p,i}$ with $v_i = c_{1,i}w_1 + \cdots + c_{p,i}w_p$ for $i = 1, \ldots, q$.

Let M be the matrix

$$M = \begin{bmatrix} c_{1,1} & c_{1,2} & c_{1,3} & \cdots & c_{1,q} \\ c_{2,1} & c_{2,2} & c_{2,3} & \cdots & c_{2,q} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{p,1} & c_{p,2} & c_{p,3} & \cdots & c_{p,q} \end{bmatrix}.$$

If A is the matrix whose columns are $v_1, \ldots, v_q$, and B the $n \times p$ matrix whose columns are the vectors $w_1, \ldots, w_p$, then the above relations can be summarized by the equation $A = BM$.

If $q > p$ then the linear lemma tells us that there is a nonzero vector in the kernel of M. But this means that there is a nonzero vector in the kernel of BM, i.e., a nonzero vector in the kernel of A.

But this contradicts the fact that the columns of A are linearly independent, since vectors in the kernel of A are exactly linear relations among the columns of A. Therefore, $q > p$ is impossible, so we must have $q \leq p$.

(b) If $v_1, \ldots, v_q$ is a basis of V, then by definition the vectors $v_1, \ldots, v_q$ are linearly independent and span V. If $w_1, \ldots, w_p$ is another basis of V, then they have the same properties.

Since the v's are linearly independent, and the w's span, part (a) tells us that $q \leq p$. On the other hand, the w's are also linearly independent, and that v's also span, so part (a) (with the roles of v and w reversed) tells us that $p \leq q$. Combining both inequalities gives $p = q$.

Therefore the number of vectors in any basis of V is the same, regardless of which basis we choose, and so $\dim(V)$ is well defined.

2. The argument we had in class had two steps. The first was based on the following idea, which is useful enough that we should give it its own name.

Proposition — If $v_1, \ldots, v_r$ are linearly independent vectors in a subspace V, and v_{r+1} is any vector outside the span of $v_1, \ldots, v_r$, then the larger set of vectors $v_1, \ldots, v_r$, v_{r+1} are also linearly independent.

Just to be thorough, let's recall the proof of this proposition:

Proof — Suppose that there is a linear relation

$$c_1v_1 + c_2v_2 + \cdots + c_rv_r + c_{r+1}v_{r+1} = \vec{0}.$$

If $c_{r+1} \neq 0$, then dividing by c_{r+1} we have:

$$v_{r+1} = \frac{-1}{c_{r+1}}(c_1v_1 + c_2v_2 + \cdots + c_rv_r),$$

contradicting the assumption that v_{r+1} is outside the span of $v_1, \ldots, v_r$.

On the other hand, if $c_{r+1} = 0$, then the above linear relation becomes

$$c_1v_1 + c_2v_2 + \cdots + c_rv_r = \vec{0},$$

which contradicts the assumption that $v_1, \ldots, v_r$ are linearly independent unless $c_1, \ldots, c_r$ are zero.

Therefore $c_1, \ldots, c_r, c_{r+1}$ are all zero, and so $v_1, \ldots, v_{r+1}$ are linearly independent. (End of proof.)

If V is a subspace of $\mathbb{R}^n$, then the argument that V has a basis proceeded like this:

If V is not the zero subspace, then pick any nonzero vector v_1 in V. If this spans V, we're done. If not, pick any nonzero vector v_2 in V outside the span of v_1. By the proposition v_1 and v_2 are linearly independent. If v_1 and v_2 span V, then we're done. If not, then pick v_3 in V outside their span. By the proposition v_1, v_2, and v_3 are linearly independent. We continue in this way: at each stage either we've found a linearly independent set which also spans V, or we pick something in V outside the span and add it to the list, using the proposition to ensure that the new list also consists of linearly independent vectors.

We will have shown the existence of a basis for V if we can only ensure that this process must stop eventually. The alternative would be that we can find an arbitrarily large set of vectors in V which are linearly independent. But, if we have any $n + 1$ vectors

$v_1, \ldots, v_{n+1}$ in V (which is in $\mathbb{R}^n$) then the matrix M with column vectors the v's is a $n \times (n+1)$ matrix. By the linear lemma, this has a nonzero vector in the kernel, and so the v's are linearly dependent.

So – there is a limit on how many linearly independent vectors we can have in V, and so the process must stop eventually, i.e., V must have a basis.

The argument actually establishes more: If we have any set $v_1, \ldots, v_r$ of linearly independent vectors in V, then this set may be extended to a basis. To see this, we just use the same argument: If $v_1, \ldots, v_r$ don't span V, pick any vector in V outside their span and add it to the list. The new set of vectors is still linearly independent thanks to the proposition.

We then just repeat this step until we finally get a set that spans. This must happen eventually, since the alternative, that there is an arbitrarily large set of vectors in V which are linearly independent, has already been ruled out.

Now suppose that W is a subspace of V. Let $w_1, \ldots, w_r$ be a basis for W. Since W is a subspace of V, these vectors are also in V, and are still linearly independent. By the above argument, the set $w_1, \ldots, w_r$ can be extended to a basis for V. Since "extending" means (possibly) adding vectors, and since the dimension of a vector space is the number of vectors in any basis, this gives $\dim(W) \leq \dim(V)$.

3. *Numbers by any other name.*

(a) If $A = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$, and $B = \begin{bmatrix} u & v \\ -v & u \end{bmatrix}$ then

$$AB = \begin{bmatrix} au - bv & av + bu \\ -av - bu & au - bv \end{bmatrix}, \quad \text{and } BA = \begin{bmatrix} au - bv & av + bu \\ -av - bu & au - bv \end{bmatrix},$$

so $AB = BA$.

(b) Since

$$\begin{bmatrix} u & v \\ -v & u \end{bmatrix} \begin{bmatrix} \frac{u}{u^2+v^2} & \frac{-v}{u^2+v^2} \\ \frac{v}{u^2+v^2} & \frac{u}{u^2+v^2} \end{bmatrix} = \begin{bmatrix} \frac{u^2+v^2}{u^2+v^2} & \frac{uv-vu}{u^2+v^2} \\ \frac{-uv+vu}{u^2+v^2} & \frac{u^2+v^2}{u^2+v^2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

as long as one of u or v isn't zero (i.e., the matrix isn't the zero matrix), $A = \begin{bmatrix} u & v \\ -v & u \end{bmatrix}$ has an inverse, and the inverse is of the same special type.

(c) If $D = \begin{bmatrix} 3 & -5 \\ 5 & 3 \end{bmatrix}$, then $D^2 = \begin{bmatrix} -16 & -30 \\ 30 & -16 \end{bmatrix}$.

(d) If $A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -11 \\ 11 & 1 \end{bmatrix}$, and $C = \begin{bmatrix} -12 & 14 \\ -14 & -12 \end{bmatrix}$, then we're trying to solve the equation

$$AX^2 + BX + C = 0,$$

where the zero above means the zero matrix.

If the matrices we're working with can be treated just like numbers, then the thing to try is the quadratic formula:

$$X = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}.$$

Since $B^2 - 4AC = \begin{bmatrix} -16 & -30 \\ 30 & -16 \end{bmatrix}$ part (c) tells us that we can use $\begin{bmatrix} 3 & -5 \\ 5 & 3 \end{bmatrix}$ for its square root.

The formula from part (b) gives $\begin{bmatrix} \frac{1}{4} & -\frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{bmatrix}$ as the inverse of $2A$, so the quadratic formula gives

$$X_1 = \frac{-B + \sqrt{B^2 - 4AC}}{2A} = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix},$$

and

$$X_2 = \frac{-B - \sqrt{B^2 - 4AC}}{2A} = \begin{bmatrix} 3 & 5 \\ -5 & 3 \end{bmatrix}.$$

Plugging these in to the equation, we have

$$AX_1^2 + BX_1 + C = \begin{bmatrix} -1 & 7 \\ -7 & -1 \end{bmatrix} + \begin{bmatrix} 13 & 21 \\ -21 & 13 \end{bmatrix} + \begin{bmatrix} -12 & 14 \\ -14 & -12 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

and

$$AX_2^2 + BX_2 + C = \begin{bmatrix} -46 & 14 \\ -14 & -46 \end{bmatrix} + \begin{bmatrix} 58 & -28 \\ 28 & 58 \end{bmatrix} + \begin{bmatrix} -12 & 14 \\ -14 & -12 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

So the quadratic formula works with these "numbers"!

Note that these are just the solutions with X in our special form. If X were allowed to be an arbitrary 2×2 matrix then it turns out that there are two other possibilities for X.

4.

(a) Sequences of the form $(a, a+k, a+2k, a+3k, \ldots)$ do form a subspace. The set is closed under addition:

$$(a_1, a_1+k_1, a_1+2k_1, a_1+3k_1, \ldots\ldots) + (a_2, a_2+k_2, a_2+2k_2, a_2+3k_2, \ldots\ldots) =$$

$$(a_1+a_2, (a_1+a_2)+(k_1+k_2), (a_1+a_2)+2(k_1+k_2), (a_1+a_2)+3(k_1+k_2), \ldots\ldots)$$

And under scalar multiplication:

$$c(a_1, a_1+k_1, a_1+2k_1, a_1+3k_1, \ldots\ldots) = (ca_1, ca_1+ck_1, ca_1+2ck_1, ca_1+3ck_1, \ldots\ldots)$$

which is another sequence of the correct type.

(b) This subset is not a subspace of the vector space of infinite sequences. It is closed under scalar multiplication, but not under addition. A simple example is

$$(1, 1, 1, 1, \ldots\ldots) + (1, 2, 4, 8, \ldots\ldots) = (2, 3, 5, 9, \ldots\ldots)$$

the first two are in the subset, but the sum is not. Since the ratio between the first and second term is $3/2$, if the sequence were in the subset, the third term would have to be $3(3/2) = 9/2$, and not 5. The other terms are similarly incorrect.

5. If we fix the matrix B (any matrix B, it doesn't have to be diagonal) then the matrices A which commute with B do form a subspace. If A_1 and A_2 are in the subspace, then

$$(A_1 + A_2)B = A_1B + A_2B = BA_1 + BA_2 = B(A_1 + A_2)$$

so $A_1 + A_2$ is in the subspace. (Note that the middle equality is because both A_1 and A_2 are in the subspace).

And, if A is in the subspace, and c any scalar, then $cAB = cBA = B(cA)$, so cA is in the subspace too.

Restricting ourselves to the case that B is a diagonal matrix, let's write A and B as

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} \\ a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} \end{bmatrix}, \text{ and } B = \begin{bmatrix} b_1 & 0 & 0 & 0 \\ 0 & b_2 & 0 & 0 \\ 0 & 0 & b_3 & 0 \\ 0 & 0 & 0 & b_4 \end{bmatrix}.$$

Then

$$AB = \begin{bmatrix} b_1a_{1,1} & b_2a_{1,2} & b_3a_{1,3} & b_4a_{1,4} \\ b_1a_{2,1} & b_2a_{2,2} & b_3a_{2,3} & b_4a_{2,4} \\ b_1a_{3,1} & b_2a_{3,2} & b_3a_{3,3} & b_4a_{3,4} \\ b_1a_{4,1} & b_2a_{4,2} & b_3a_{4,3} & b_4a_{4,4} \end{bmatrix}, \text{ and } BA = \begin{bmatrix} b_1a_{1,1} & b_1a_{1,2} & b_1a_{1,3} & b_1a_{1,4} \\ b_2a_{2,1} & b_2a_{2,2} & b_2a_{2,3} & b_2a_{2,4} \\ b_3a_{3,1} & b_3a_{3,2} & b_3a_{3,3} & b_3a_{3,4} \\ b_4a_{4,1} & b_4a_{4,2} & b_4a_{4,3} & b_4a_{4,4} \end{bmatrix}.$$

For these matrices to be equal, they have to be equal in every entry. The equation in the (i, j)-th entry is $b_i a_{i,j} = b_j a_{i,j}$ or $(b_i - b_j)a_{i,j} = 0$.

This tells us that entry $a_{i,j}$ can be nonzero if and only if $b_i = b_j$. The dimension of the vector space of matrices A which commute with B is equal to the number of possible nonzero entries of A, i.e., the dimension of this vector space is just the number of pairs (i, j) with $b_i = b_j$.

The possibilities are:

- If all four of the b_i's are equal, then the space of commuting matrices has dimension $4^2 = 16$, i.e., is all of the 4×4 matrices.
- If three of the b_i's are equal, and one different from the others, the dimension is $3^2 + 1 = 10$.
- If two of the b_i's are equal, and the other two different from these two and from each other, the dimension is $2^2 + 1 + 1 = 6$.
- If two of the b_i's are equal, and the other are equal to each other but different from the first two, then the dimension is $2^2 + 2^2 = 8$.
- If no two of the b_i's are equal, then the dimension is $1 + 1 + 1 + 1 = 4$.