1. Let $\underline{a} = (a_0,\, a_1,\, a_2,\, a_3,\, \ldots) = (3, 6, 14, 26, 98, \ldots)$ be the sequence defined in the problem. Let V be the subset of the infinite sequences $(x_0, x_1, x_2, x_3, x_4, x_5, \ldots)$ such that the x's satisfy the same recursion relations as the a's, i.e., such that $x_3 = 6x_2 - 11x_1 + 6x_0$, that $x_4 = 6x_3 - 11x_2 + 6x_1$, and that in general

$$x_{n+3} = 6x_{n+2} - 11x_{n+1} + 6x_n \ \text{ for } n \geq 0.$$

The set V is actually a *subspace* of the vector space of infinite sequences.

That is, suppose that

$$\underline{v} = (v_0, v_1, v_2, v_3, v_4, v_5, \ldots),$$

and

$$\underline{w} = (w_0, w_1, w_2, w_3, w_4, w_5, \ldots),$$

are two sequences in V (so they satisfy all the relations above). Then

$$\underline{v} + \underline{w} = (v_0 + w_0, v_1 + w_1, v_2 + w_2, v_3 + w_3, v_4 + w_4, \ldots)$$

is also in V. To see this, let z_i be the i-th coordinate of $\underline{v} + \underline{w}$, i.e., so that $z_i = v_i + w_i$ for all $i \geq 0$. To check if the sum is in V we need to see that the z's satisfy all the relations. But for any $n \geq 0$,

$$\begin{aligned}
z_{n+3} &= v_{n+3} + w_{n+3} && \text{[by definition of the } z\text{'s]} \\
&= (6v_{n+2} - 11v_{n+1} + 6v_n) + (6w_{n+2} - 11w_{n+1} + 6w_n) && \text{[since } \underline{v} \text{ and } \underline{w} \text{ are in } V\text{]} \\
&= 6(v_{n+2} + w_{n+2}) - 11(v_{n+1} + w_{n+1}) + 6(v_n + w_n) && \text{[rearranging terms]} \\
&= 6z_{n+2} - 11z_{n+1} + 6z_n && \text{[using the def. of } z\text{'s again]}
\end{aligned}$$

So the sequence of z's is also in V. Similarly, if c is any number, then

$$c\underline{v} = (cv_0, cv_1, cv_2, cv_3, cv_4, \ldots)$$

is also in V. The argument is even easier: writing z_i for the i-th term of $c\underline{v}$ (so that this time $z_i = cv_i$), we can see that

$$\begin{aligned}
z_{n+3} &= cv_{n+3} && \text{[by definition of the } z\text{'s]} \\
&= c(6v_{n+2} - 11v_{n+1} + 6v_n) && \text{[since } \underline{v} \text{ is in } V\text{]} \\
&= 6(cv_{n+2}) - 11(cv_{n+1}) + 6(cv_n) && \text{[Rearranging terms.]} \\
&= 6z_{n+2} - 11z_{n+1} + 6z_n && \text{[using the def. of } z\text{'s again]}
\end{aligned}$$

So V is also closed under multiplication by scalars. Therefore V is a subspace of the space of all infinite sequences.

What is the dimension of V? The recursion equation gives a good clue: if the first three terms of any sequence in V are zero, then the recursion equation says that all the rest are zero too, i.e., the only sequence in V with the first three terms equal to zero is the zero vector.

That means that if we take the three sequences

$$\begin{aligned} \underline{e}_1 &= (1, 0, 0, 6, 36, 150, 540, 1806, \ldots) \\ \underline{e}_2 &= (0, 1, 0, -11, -60, -239, -840, -2771, \ldots) \\ \underline{e}_2 &= (0, 0, 1, 6, 25, 90, 301, 966, \ldots) \end{aligned}$$

constructed by starting off with $(1, 0, 0, \ldots)$, $(0, 1, 0, \ldots)$, and $(0, 0, 1, \ldots)$ and using the rules to determine the rest of the sequence, that these three must span V. Given any other sequence in V, say $\underline{v} = (a, b, c, d, e, f, \ldots)$, we see that $a\underline{e}_1 + b\underline{e}_2 + c\underline{e}_3$ has first entries $(a, b, c, \ldots)$, and hence $\underline{v} - a\underline{e}_1 - b\underline{e}_2 - c\underline{e}_3$ is a sequence in V whose first three entries are zero, and so must be the zero vector, i.e., $\underline{v} = a\underline{e}_1 + b\underline{e}_2 + c\underline{e}_3$.

This shows that $\underline{e}_1$, $\underline{e}_2$, and $\underline{e}_3$ span V. Since they are also linearly independent, they form a basis for V, and so V is three dimensional.

The sequence $\underline{a}$ that we're looking at is the linear combination $\underline{a} = 3\underline{e}_1 + 6\underline{e}_2 + 14\underline{e}_3$, but this doesn't really help us compute a general formula for the n-th entry of the $\underline{a}$ sequence, since we don't have a good way of computing the n-the entry of the basis vectors $\underline{e}_1$, $\underline{e}_2$, and $\underline{e}_3$.

Instead, we look for a different basis, one better suited to finding the n-th term.

Let's look for sequences in V of the form

$$(1, \alpha, \alpha^2, \alpha^3, \alpha^4, \alpha^5, \ldots\ldots),$$

the n-the term is easy to compute – it's $x_n = \alpha^n$.

When is such a sequence in V? It has to satisfy all the relations, so for instance we need $x_3 = 6x_2 - 11x_1 + 6x_0$ or $\alpha^3 = 6\alpha^2 - 11\alpha + 6$, and that $x_4 = 6x_3 - 11x_2 + 6x_1$ or $\alpha^4 = 6\alpha^3 - 11\alpha^2 + 6\alpha$, or in general $\alpha^{n+3} = 6\alpha^{n+2} - 11\alpha^{n+1} + 6\alpha^n$.

But all of these equations are implied by the first one: $\alpha^3 = 6\alpha^2 - 11\alpha + 6$, since multiplying this by α^n gives the general relation. So, the condition really becomes, α is a root of $x^3 = 6x^2 - 11x + 6$ or $x^3 - 6x^2 + 11x - 6$.

Factoring, we get $x^3 - 6x^2 + 11x - 6 = (x - 1)(x - 2)(x - 3)$ so there are three such possible α's: $\alpha_1 = 1$, $\alpha_2 = 2$, and $\alpha_3 = 3$.

Letting

$$\begin{aligned}
\underline{v}_1 &= (1, 1, 1, 1, 1, 1, 1, 1, \ldots) \\
\underline{v}_2 &= (1, 2, 2^2, 2^3, 2^4, 2^5, 2^6, 2^7, \ldots) \\
\underline{v}_2 &= (1, 3, 3^2, 3^3, 3^4, 3^5, 3^6, 3^7, \ldots)
\end{aligned}$$

we try and write $\underline{a}$ in terms of $\underline{v}_1$, $\underline{v}_2$, and $\underline{v}_3$, i.e., we look for coefficients c_1, c_2 and c_3 with $\underline{a} = c_1\underline{v}_1 + c_2\underline{v}_2 + c_3\underline{v}_3$. Looking at the first three entries of $\underline{a}$, this leads to the equations

$$\begin{aligned}
c_1 + c_2 + c_3 &= 3 \\
c_1 + 2c_2 + 3c_3 &= 6 \\
c_1 + 4c_2 + 9c_3 &= 14,
\end{aligned}$$

which has solutions $c_1 = 1$, $c_2 = 1$, and $c_3 = 1$. This gives $\underline{a} = \underline{v}_1 + \underline{v}_2 + \underline{v}_3$, or (by looking at the n-th entries of the $\underline{v}$'s) that $a_n = 1^n + 2^n + 3^n$.

2. *Polynomial interpolation.*

(a) The vector space P_5 of polynomials of degree ≤ 5 is six dimensional. One basis is 1, x, x^2, x^3, x^4, and x^5.

(b) We showed in class that given the vector space V of all possible functions (on $\mathbb{R}$), the map $V \longrightarrow \mathbb{R}$ given by sending f to $f(x_i)$, i.e., by plugging an x_i is a linear map. (This actually followed from the definition of sum of functions, if you remember).

This means that for any numbers x_1, x_2, $\ldots$, x_6 the map from the vector space V to $\mathbb{R}^6$ given by $f \mapsto (f(x_1), f(x_2), f(x_3), f(x_4), f(x_5), f(x_6))$ is also a linear map, since checking linearity amounts to checking what happens in each of the coordinates separately, and the fact that the map is linear in each coordinate separately is exactly the fact mentioned above.

Since this map is linear on the space V of all functions from $\mathbb{R}$ to $\mathbb{R}$, it is also linear when restricted to the subspace P_5 of polynomials of degree ≤ 5.

(c) Saying that $T(p) = (y_1, y_2, \ldots, y_6)$ is the same as saying that $p(x_1) = y_1$, $p(x_2) = y_2$, $\ldots$, $p(x_6) = y_6$, in other words, that p is one of the polynomials we're looking for. Since we want p to be unique we need $\ker(T) = 0$. Since we want to be able to find a p for any choice of y_1, $\ldots$, y_6, we need T to be surjective, i.e., that $\operatorname{im}(T) = \mathbb{R}^6$.

(d) Since $\dim(P_5) = 6$, the rank-nullity theorem says that $\dim(\ker(T)) + \dim(\operatorname{im}(T)) = 6$. So, if $\ker(T) = 0$, this means that $\dim(\ker(T)) = 0$ and the rank-nullity theorem then implies that $\dim(\operatorname{im}(T)) = 6$ or that T is surjective.

(e) If p is in $\ker(T)$ then $T(p) = (0, 0, 0, \dots, 0)$ (by definition of the kernel) but this means (by the definition of T) that $p(x_1) = 0$, $p(x_2) = 0$, $\dots$, $p(x_6) = 0$.

If $p(x_i) = 0$, this means that $(x - x_i)$ is a factor of p. Since a nonzero polynomial of degree ≤ 5 can't have more than 5 factors, this means that the only polynomial in the kernel of T is the zero polynomial, i.e., $\ker(T) = 0$, and so (reversing the chain of implications above) there is a unique p with $p(x_1) = y_1$, $\dots$, $p(x_6) = y_6$.

3. Letting V be the set of linear transformations from V_1 to V_2, we're given two proposed operations on V, and we want to see if this makes V into a vector space. The issue is whether or not the result of the operations are still elements of V, i.e., still linear transformations from V_1 to V_2.

From the descriptions, it's pretty clear that the $T_1 + T_2$ and cT are still maps from V_1 to V_2, the thing we really need to check is that they're still linear transformations.

(a) For any f and g in V_1, then

$$\begin{aligned}
(T_1 + T_2)(f + g) &= T_1(f + g) + T_2(f + g) && \text{[by the def. of } T_1 + T_2\text{]} \\
&= T_1(f) + T_1(g) + T_2(f) + T_2(g) && \text{[since } T_1 \text{ and } T_2 \text{ are linear]} \\
&= (T_1(f) + T_2(f)) + (T_2(f) + T_2(g)) && \text{[rearranging terms]} \\
&= (T_1 + T_2)(f) + (T_1 + T_2)(g) && \text{[by def. of } T_1 + T_2 \text{ again]}
\end{aligned}$$

On the other hand, for any f in V_1 and any scalar c,

$$\begin{aligned}
(T_1 + T_2)(cf) &= T_1(cf) + T_2(cf) && \text{[by the def. of } T_1 + T_2\text{]} \\
&= cT_1(f) + cT_2(f) && \text{[since } T_1 \text{ and } T_2 \text{ are linear]} \\
&= c(T_1(f) + T_2(f)) && \text{[rearranging terms]} \\
&= c((T_1 + T_2)(f)) && \text{[by def. of } T_1 + T_2 \text{ again]}
\end{aligned}$$

so $T_1 + T_2$ is a linear transformation from V_1 to V_2.

(b) For any f and g in V_1,

$$\begin{aligned}
(cT)(f + g) &= c(T(f + g)) && \text{[by the def. of } cT\text{]} \\
&= c(T(f) + T(g)) && \text{[since } T \text{ is linear]} \\
&= c(T(f)) + c(T(g)) && \text{[rearranging terms]} \\
&= (cT)(f) + (cT)(g) && \text{[by def. of } cT \text{ again.]}
\end{aligned}$$

Similarly, for any f in V_1, and any scalar c' (to differentiate it from c),

$$\begin{aligned}(cT)(c'f) &= c(T(c'f)) && \text{[by the def. of } cT\text{]}\\ &= c(c'T(f)) && \text{[since } T \text{ is linear]}\\ &= cc'(T(f)) && \text{[rearranging terms]}\\ &= c'((cT)(f)) && \text{[by def. of } cT \text{ again.]}\end{aligned}$$

So cT is a linear transformation from V_1 to V_2.

(c) Once we've chosen a basis for $\mathbb{R}^2$ and $\mathbb{R}^3$, each linear map can be represented by a matrix, and the operations above are just addition and scalar multiplication of matrices. So the vector space of linear maps from $\mathbb{R}^3$ to $\mathbb{R}^2$ is the same as the vector space of 2×3 matrices, which is 6 dimensional.

4.

(a) The most elegant solution is probably the following: the vector space of $n \times n$ matrices is n^2 dimensional. The matrices A^{n^2}, A^{n^2-1}, A^{n^2-2}, $\ldots A^2$, A, I_n are n^2+1 elements of this n^2 dimensional vector space, so they must be linearly dependent. I.e., there must exist c_{n^2}, c_{n^2-1}, c_{n^2-2}, $\ldots$, c_2, c_1, c_0, not all zero, with

$$c_{n^2}A^{n^2} + c_{n^2-1}A^{n^2-1} + \cdots + c_2A^2 + c_1A + c_0I_n = 0.$$

Letting $p(x)$ be the polynomial

$$p(x) = c_{n^2}x^{n^2} + c_{n^2-1}x^{n^2-1} + \cdots + c_2x^2 + c_1x + c_0$$

defined by the same coefficients c_i solves the problem.

(b) $p(x) = x^2 - 7x + 10$ works, since

$$\begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}^2 - 7\begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix} + 10\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 11 & 14 \\ 7 & 18 \end{bmatrix} - \begin{bmatrix} 21 & 14 \\ 7 & 28 \end{bmatrix} + \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$