1. For the basis $\beta = (3,5),(1,2)$ in $\mathbb{R}^2$, the "from β coordinates to usual coordinates" matrix is $M = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}$, while the "from standard to β" matrix is its inverse $M^{-1} = \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}$.

(a) So, the vector $(4,3)$ in usual coordinates is the same as $(5,1)_\beta$ in β coordinates since

$$\begin{bmatrix} 2 & -1 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}.$$

Similarly, since

$$\begin{aligned} \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \text{ and} \\ \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix} &= \begin{bmatrix} -1 \\ 4 \end{bmatrix}. \end{aligned}$$

we have $(1,2) = (0,1)_\beta$ and $(1,3) = (-1,4)_\beta$. We could have seen the second formula without any calculation. Since $(1,2)$ is the second basis vector in the basis β, it's β-coordinate is necessarily $(0,1)_\beta$.

(b) Going the other way is even easier:

$$\begin{aligned} \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \end{bmatrix} &= \begin{bmatrix} 6 \\ 10 \end{bmatrix} \\ \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} &= \begin{bmatrix} 8 \\ 3 \end{bmatrix} \\ \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} -1 \\ 4 \end{bmatrix} &= \begin{bmatrix} 1 \\ 3 \end{bmatrix} \end{aligned}$$

So $(2,0)_\beta = (6,10)$, $(1,1)_\beta = (8,3)$, and $(-1,4)_\beta = (1,3)$. We also found the last equality in part (a), just the other way around.

2. If $\beta = (1,1,1)$, $(2,0,1)$, and $(3,2,1)$ is a new basis in $\mathbb{R}^3$, the matrix changing from β coordinates to usual coordinates is

$$N = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix}.$$

If $\alpha = (3,5)$, $(1,2)$ is a new basis in $\mathbb{R}^2$ then the matrix changing from the usual coordinates to α coordinates is

$$M^{-1} = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}$$

as computed in the previous question.

The matrix for T in the new coordinates is then

$$M^{-1}AN = \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix} \begin{bmatrix} 2 & 1 & 5 \\ 1 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 11 & 13 & 18 \\ -25 & -30 & -41 \end{bmatrix}.$$

3.

(a) In the new basis $v_1 = (1,0)_\beta$ and $v_2 = (0,1)_\beta$. We're looking for the matrix that does $T((1,0)_\beta) = (1,0)_\beta$ and $T((0,1)_\beta) = (0,0)_\beta$, and this is given by

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}_\beta.$$

(b) The matrix changing from β coordinates to standard coordinates is $N = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, its inverse $N^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$ changes from standard coordinates to β coordinates.

The matrix for T in standard coordinates is then

$$NBN^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

This is the same as the matrix for $m = 1$ in problem 4, homework 3, since they're the same linear transformation: they send the vector $(1,1)$ (a vector on the line of slope 1) to itself, and send the vector $(1,-1)$ (a vector perpendicular to the line) to the zero vector.

4. Letting V be the vector space, and $\beta = \sin(x), \cos(x)$ the basis, we have that $\sin(x) = (1,0)_\beta$ and $\cos(x) = (0,1)_\beta$ in β coordinates.

Since the derivative map T operates on $\sin(x)$ and $\cos(x)$ by $T(\sin(x)) = \cos(x)$ and $T(\cos(x)) = -\sin(x)$, this is the same as

$$T((1,0)_\beta) = (0,1)_\beta$$

and

$$T((0,1)_\beta) = (-1,0)_\beta,$$

and so the matrix for T in this basis is given by $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.