1. The expressions are

(a) $\begin{bmatrix} 3 & 5 & 8 \\ 1 & 2 & -4 \end{bmatrix} + \begin{bmatrix} 4 & 6 & 0 \\ 0 & -3 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 11 & 8 \\ 1 & -1 & -2 \end{bmatrix}$

(b) $3\begin{bmatrix} 1 & -2 \\ -6 & 5 \end{bmatrix} = \begin{bmatrix} 3 & -6 \\ -18 & 15 \end{bmatrix}$

(c) $4\begin{bmatrix} 1 & 0 & 4 \\ 2 & 7 & 3 \end{bmatrix} - 3\begin{bmatrix} 2 & 1 & 1 \\ 7 & 0 & -2 \end{bmatrix} = \begin{bmatrix} -2 & -3 & 13 \\ -15 & 21 & 18 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & -2 & 5 \\ 2 & 3 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 1\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 2\begin{bmatrix} -2 \\ 3 \end{bmatrix} + 3\begin{bmatrix} 5 \\ 4 \end{bmatrix} = \begin{bmatrix} 12 \\ 20 \end{bmatrix}$

2. Each of these problems can be converted into a problem about solving linear equations, and we know how to solve linear equations. The general strategy is to write down an augmented matrix which represents the linear equations we're trying to solve, put the matrix into RREF, and then read off the solutions.

(a) $\left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 2 & 3 \\ 2 & 0 & 3 & 10 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right]$

Here there is a unique solution, the only way to write $\vec{b}$ as a linear combination of $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$, is $2\vec{v}_1 - \vec{v}_2 + 2\vec{v}_3 = \vec{b}$.

(b) $\left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 2 & 3 \\ 2 & 0 & -4 & 10 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$

Here there is no solution – there is no way to write $\vec{b}$ as a linear combination of $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$.

(c) $\left[\begin{array}{ccc|c} 1 & 1 & 0 & 8 \\ 0 & 1 & 2 & 5 \\ 2 & 0 & -4 & 6 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & -2 & 3 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right]$

Finally, this system of equations has infinitely many solutions, a one parameter family. There are lots of ways to write $\vec{b}$ as a linear combination of $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$, and the general solution is

$$(3+t)\vec{v}_1 + (5-2t)\vec{v}_2 + t\vec{v}_3 = \vec{b}.$$

3. If we look at the augmented matrix $\left[A \mid \vec{b}\right]$, then the condition that the equation $A\vec{x} = \vec{b}$ has a unique solution is telling us that the RREF of the augmented matrix looks like

$$\text{RREF}\left(\left[A \mid \vec{b}\right]\right) = \left[\begin{array}{ccc|c} 1 & 0 & 0 & u \\ 0 & 1 & 0 & v \\ 0 & 0 & 1 & w \\ 0 & 0 & 0 & 0 \end{array}\right],$$

where u, v, and w are numbers depending on $\vec{b}$ (they're the unique solutions to $A\vec{x} = \vec{b}$).

If we instead look at the equation $A\vec{x} = \vec{c}$, we should similarly look at the augmented matrix $[A \mid \vec{c}]$. We already (from above) know what happens to the first three columns as we put the matrix into RREF. The conclusion is that the RREF of the augmented matrix must look like this:

$$\text{RREF}\left(\left[A \mid \vec{c}\right]\right) = \left[\begin{array}{ccc|c} 1 & 0 & 0 & i \\ 0 & 1 & 0 & j \\ 0 & 0 & 1 & k \\ 0 & 0 & 0 & l \end{array}\right],$$

for some numbers i, j, k, and l.

From this form we can conclude that if $l \neq 0$ then there is no solution to the equation, while if $l = 0$ then there is a unique solution.

Therefore, there is either no solution or a unique solution to the equation $A\vec{x} = \vec{c}$. Which possibility occurs depends on which vector $\vec{c}$ in $\mathbb{R}^4$ we pick.

4. The principle that we're using is that the linear relations among the columns of A are exactly the same as the linear relations among the columns of the RREF of A.

So, even without knowing what $\vec{v}_1, \ldots, \vec{v}_4$ are, we still know what all the linear relations are between them – we just have to look at RREF(A).

(a) The vector $(0, 1, 0)$ is not a linear combination of the vectors $(1, 0, 0)$ and $(2, 0, 0)$, and therefore (by the principle) the vector $\vec{v}_3$ is not a linear combination of $\vec{v}_1$ and $\vec{v}_2$. (and, in fact, we can see that $\vec{v}_2$ is twice $\vec{v}_1$).

(b) The vector $(1, 4, 0)$ is a linear combination of $(1, 0, 0)$ and $(0, 1, 0)$ in exactly one way: $(1, 4, 0) = 1 \cdot (1, 0, 0) + 4 \cdot (0, 1, 0)$, and so $\vec{v}_4$ is a linear combination of $\vec{v}_1$ and $\vec{v}_3$ in exactly one way: $\vec{v}_4 = \vec{v}_1 + 4\vec{v}_3$.

(c) Rewriting the previous equation, we have $\vec{v}_3 = \frac{1}{4}(\vec{v}_4 - \vec{v}_1) = \frac{1}{4}\vec{v}_4 - \frac{1}{4}\vec{v}_1$. This expression is also unique (since the previous one was). We could also arrive at this equation by noticing that $(0, 1, 0) = \frac{1}{4}((1, 4, 0) - (1, 0, 0))$, from the RREF, and then translate this back to the relation among the columns of A.

5. The notation in this question may have confused several people. Usually we've been using x_1 to mean a single number, and if we have a vector like $\vec{x}$, we think of it as having components $(x_1, x_2, \ldots, x_m)$.

In this case however, $\vec{x}_1$ is itself a vector, and so represents several numbers, something like $\vec{x}_1 = (x_{1,1}, x_{1,2}, \ldots, x_{1,n})$ for want of a better way to describe them. Similarly, the symbol $\vec{x}_h$ is a vector, perhaps with components $\vec{x}_h = (x_{h,1}, x_{h,2}, \ldots, x_{h,n})$.

If A is an $m \times n$ matrix, which we can think of as being made up of n column vectors, each in $\mathbb{R}^m$,

$$A = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{bmatrix},$$

then the expression $A\vec{x}_1$ has the meaning $x_{1,1}\vec{v}_1 + x_{1,2}\vec{v}_2 + \cdots x_{1,n}\vec{v}_n$, the linear combination of the columns of A with coefficients $x_{1,1}, \ldots,\ x_{1,m}$. Similarly, the expression $A\vec{x}_h$ means the linear combination $x_{h,1}\vec{v}_1 + x_{h,2}\vec{v}_2 + \cdots x_{h,n}\vec{v}_n$, of the columns of A with coefficients $x_{h,1}, \ldots,\ x_{h,n}$.

Expressions like $\vec{x}_1 + \vec{x}_h$ mean as always that we add coordinate by coordinate $\vec{x}_1 + \vec{x}_h = (x_{1,1} + x_{1,h}, x_{1,2} + x_{h,2}, \ldots, x_{1,n} + x_{h,n})$.

The vector $\vec{b}$ is some vector in $\mathbb{R}^m$, and we're given that $A\vec{x}_1 = \vec{b}$.

(a) If $A\vec{x}_h = \vec{0}$, then $A(\vec{x}_1 + \vec{x}_h) = A\vec{x}_1 + A\vec{x}_h = \vec{b} + \vec{0} = \vec{b}$.

If that answer seems a bit too slick, it's worth writing out the identity $A(\vec{x}_1 + \vec{x}_h) = A\vec{x}_1 + A\vec{x}_h$ in terms of the components to see what it's saying:

$$\begin{aligned} A(\vec{x}_1 + \vec{x}_h) &= \\ &= (x_{1,1} + x_{h,1})\vec{v}_1 + (x_{1,2} + x_{h,2})\vec{v}_2 + \cdots + (x_{1,n} + x_{h,n})\vec{v}_n \\ &= (x_{1,1}\vec{v}_1 + x_{1,2}\vec{v}_2 + \cdots + x_{1,n}\vec{v}_n) + (x_{h,1}\vec{v}_1 + x_{h,2}\vec{v}_2 + \cdots + x_{h,n}\vec{v}_n) \\ &= \vec{b} + \vec{0} \\ &= \vec{b}. \end{aligned}$$

(b) Similarly, if $A\vec{x}_2 = \vec{b}$ then

$$A(\vec{x}_2 - \vec{x}_1) = A\vec{x}_2 - A\vec{x}_1 = \vec{b} - \vec{b} = \vec{0},$$

so $\vec{x}_2 - \vec{x}_1$ is also a solution of the equation $A\vec{x} = \vec{0}$. It's worth writing this out in terms of the components too, just to make sure that it makes sense.

(c) What parts (a) and (b) together are saying is that every solution of the equation $A\vec{x} = \vec{b}$ is the solution $\vec{x}_1$ translated by some element of the solutions to $A\vec{x} = \vec{0}$.

Applying this to the problem, to see all the solutions to $A\vec{x} = \vec{b}$, we just have to take the solutions to $A\vec{x} = \vec{0}$ and translate them by $\vec{x}_1$: