1.

(a) $$\begin{bmatrix} 1 & -2 \\ 3 & 1 \\ 4 & -7 \end{bmatrix} \begin{bmatrix} 1 & -3 & -3 \\ 2 & 1 & 2 \end{bmatrix} = \begin{bmatrix} -3 & -5 & -7 \\ 5 & -8 & -7 \\ -10 & -19 & -26 \end{bmatrix}$$

(b) $$\begin{bmatrix} 1 & -3 & -3 \\ 2 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 3 & 1 \\ 4 & -7 \end{bmatrix} = \begin{bmatrix} -20 & 16 \\ 13 & -17 \end{bmatrix}$$

(c) $$\begin{bmatrix} -1 & 3 \\ 4 & 6 \end{bmatrix} \begin{bmatrix} 5 & -2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ 26 & -14 \end{bmatrix}$$

(d) $$\begin{bmatrix} 1 & 2 & 3 \\ -2 & -3 & 1 \\ 4 & 5 & 2 \end{bmatrix} \begin{bmatrix} 4 & 2 & 1 \\ 3 & 3 & 3 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 10 & 11 & 13 \\ -17 & -12 & -9 \\ 31 & 25 & 23 \end{bmatrix}$$

(e) $$\left[\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \left[\begin{array}{cc|cc} 1 & 2 & 2 & 3 \\ 3 & 4 & 4 & 5 \\ \hline 0 & 0 & 1 & 2 \\ 0 & 0 & 3 & 4 \end{array}\right] = \begin{bmatrix} 1 & 2 & 5 & 7 \\ 3 & 4 & 5 & 7 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 3 & 4 \end{bmatrix}$$

2. It might be a bit easier to use the variables u and v for the transformation T_2, in order to try and make the notation a bit clearer. We have

$$T_1(x, y, z) = \begin{bmatrix} 3x + 7z \\ x + 2y + 8z \end{bmatrix}, \text{ and } T_2(u, v) = \begin{bmatrix} u + 4v \\ 2u + 3v \\ -u + 2v \end{bmatrix},$$

and so

(a)

$$\begin{aligned} T_3(x, y, z) &= T_2(T_1(x, y, z)) \\ &= T_2(3x + 7z,\ x + 2y + 8z) = \begin{bmatrix} (3x + 7z) + 4(x + 2y + 8z) \\ 2(3x + 7z) + 3(x + 2y + 8z) \\ -(3x + 7z) + 2(x + 2y + 8z) \end{bmatrix}, \\ &= \begin{bmatrix} 7x + 8y + 39z \\ 9x + 6y + 38z \\ -x + 4y + 9z \end{bmatrix}. \end{aligned}$$

(b) Plugging in the vectors $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$ into the formulas, we see that

$$T_3(1,0,0) = \begin{bmatrix} 7 \\ 9 \\ -1 \end{bmatrix}, \quad T_3(0,1,0) = \begin{bmatrix} 8 \\ 6 \\ 4 \end{bmatrix}, \quad \text{and } T_3(0,0,1) = \begin{bmatrix} 39 \\ 38 \\ 9 \end{bmatrix}.$$

and so the matrix for T_3 is $C = \begin{bmatrix} 7 & 8 & 39 \\ 9 & 6 & 38 \\ -1 & 4 & 9 \end{bmatrix}$.

(c) Similarly, using the formulas for T_1 we get

$$T_1(1,0,0) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \quad T_1(0,1,0) = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \quad \text{and } T_1(0,0,1) = \begin{bmatrix} 7 \\ 8 \end{bmatrix},$$

so the matrix for T_1 is $A = \begin{bmatrix} 3 & 0 & 7 \\ 1 & 2 & 8 \end{bmatrix}$, while for T_2 we have

$$T_2(1,0) = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \text{ and } T_2(0,1) = \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix}$$

giving the matrix $B = \begin{bmatrix} 1 & 4 \\ 2 & 3 \\ -1 & 2 \end{bmatrix}$.

(d) Multiplying, we have

$$\begin{bmatrix} 1 & 4 \\ 2 & 3 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 3 & 0 & 7 \\ 1 & 2 & 8 \end{bmatrix} = \begin{bmatrix} 7 & 8 & 39 \\ 9 & 6 & 38 \\ -1 & 4 & 9 \end{bmatrix}$$

as expected. (You should show the details of this multiplication – I'm skipping them).

3. In order for a matrix to be invertible, it should be square (say $n \times n$) and have rank n, i.e., if we put it in RREF, we should get the $n \times n$ identity matrix I_n. We saw that this is exactly the condition so that the linear transformation described by the matrix was invertible as a function.

(a) This matrix is pretty clearly invertible: we could either row reduce it, or even, thinking geometrically, see that this transformation stretches the x-axes by a factor of 2 and the y-axis by a factor of 3. Its inverse is, from either point of view, the matrix

$$\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{bmatrix}.$$

(b) This matrix can not be invertible; it's not even square.

(c) This matrix is invertible. Row reducing:

$$\left[\begin{array}{cc|cc} 5 & 7 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{cc|cc} 1 & 0 & 4 & -7 \\ 0 & 1 & -3 & 5 \end{array}\right]$$

we see that the inverse is $\begin{bmatrix} 4 & -7 \\ -3 & 5 \end{bmatrix}$

(d) This is matrix is not invertible, it has rank 1. If we row reduce, we get

$$\begin{bmatrix} 2 & 1 \\ 10 & 5 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}$$

(e) This matrix isn't invertible. Its RREF is

$$\begin{bmatrix} 1 & 0 & \frac{17}{2} \\ 0 & 1 & -11 \\ 0 & 0 & 0 \end{bmatrix}.$$

and so it has rank 2. In fact, there was no need to even compute the RREF, from the form of the original matrix we could see that the RREF would continue to have a bottom row which is all zero, and so the matrix can't have rank 3.

(f) This matrix is invertible. Row reducing, we get

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 5 & 6 & 1 & 0 & 0 & 0 & 1 & 0 \\ 7 & 10 & 4 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -3 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 13 & -6 & 1 & 0 \\ 0 & 0 & 0 & 1 & -29 & 14 & -4 & 1 \end{array}\right],$$

so that the inverse matrix is $\begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 1 & 0 & 0 \\ 13 & -6 & 1 & 0 \\ -29 & 14 & -4 & 1 \end{bmatrix}$.

4. We're starting with the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ -2 & 1 & 7 \\ 2 & 2 & 1 \end{bmatrix}$.

(a) We can check that A is invertible, and find the inverse at the same time, by row-reducing:

$$\left[\begin{array}{rrr|rrr} 1 & 2 & 3 & 1 & 0 & 0 \\ -2 & 1 & 7 & 0 & 1 & 0 \\ 2 & 2 & 1 & 0 & 0 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & -13 & 4 & 11 \\ 0 & 1 & 0 & 16 & -5 & -13 \\ 0 & 0 & 1 & -6 & 2 & 5 \end{array}\right]$$

and so the inverse of A is the matrix $B = \begin{bmatrix} -13 & 4 & 11 \\ 16 & 5 & -13 \\ -6 & 2 & 5 \end{bmatrix}$.

(b) If T is the linear map $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ corresponding to the matrix A, solving the system of equations is the same as finding those vectors (x, y, z) with $T(x, y, z) = (a, b, c)$. Since the transformation T is invertible (its matrix A is invertible), we know that there is a unique solution (x, y, z) for each (a, b, c) in $\mathbb{R}^3$.

Alternatively, since the RREF of A is the identity matrix I_3, the usual argument with the row reduced form shows us that there is a unique solution. This is of course really the same argument as the one above.

(c) The definition of the inverse transformation T^{-1} is that it undoes what T does, so that for any vector (a, b, c), $T^{-1}(a, b, c)$ is exactly the vector (x, y, z) such that $T(x, y, z) = (a, b, c)$. Since we already know the matrix B for T^{-1}, we can use this to compute (x, y, z) in terms of (a, b, c):

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -13 & 4 & 11 \\ 16 & 5 & -13 \\ -6 & 2 & 5 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -13a + 4b + 11c \\ 16a + 5b - 13c \\ -6a + 2b + 5c \end{bmatrix}.$$

5.

(a) The RREF is: $\left[\begin{array}{rr|rrr} 9 & 5 & 1 & 3 & 2 \\ 5 & 3 & 1 & 1 & 3 \end{array}\right] \rightsquigarrow \left[\begin{array}{rr|rrr} 1 & 0 & -1 & 2 & -\frac{9}{2} \\ 0 & 1 & 2 & -3 & \frac{17}{2} \end{array}\right]$.

(b) If we want to solve the system $A\vec{x} = \vec{b}_1$ for $\vec{b}_1 = (1, 1)$, the method we know is to write down the augmented matrix and put it into RREF:

$$\left[\begin{array}{rr|r} 9 & 5 & 1 \\ 5 & 3 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{rr|r} 1 & 0 & -1 \\ 0 & 1 & 2 \end{array}\right].$$

and so see that $(x, y) = (-1, 2)$ is the solution to the equations $A\vec{x} = \vec{b}_1$.

We could do the same with $\vec{b}_2$ and $\vec{b}_3$, but there's no need: the answer from part (a) contains exactly all these row reductions simultaneously; the vectors $\vec{b}_1$, $\vec{b}_2$, and $\vec{b}_3$ are exactly the column vectors after the dots in part (a), and the matrix A is exactly the 2×2 matrix before the dots.

Therefore, from part (a), we can read off that the unique solution to $A\vec{x} = \vec{b}_2$ is $(x, y) = (2, -3)$, and for $A\vec{x} = \vec{b}_3$ is $(x, y) = (-\frac{9}{2}, \frac{17}{2})$.

(c) Suppose that T is the linear transformation given by A. Since A is an invertible matrix, T is an invertible transformation, and so it has an inverse transformation T^{-1} which undoes what T does.

The matrix for T^{-1} is just the inverse matrix B for A. Since T^{-1} undoes what T does, the vector $T^{-1}(\vec{e}_1)$ is the unique vector $\vec{v}_1$ so that $T(\vec{v}_1) = \vec{e}_1$. Similarly, $T^{-1}(\vec{e}_2)$ is the unique vector $\vec{v}_2$ so that $T(\vec{v}_2) = \vec{e}_1$.

Since the columns of B are exactly the vectors $T^{-1}(\vec{e}_1)$ and $T^{-1}(\vec{e}_2)$ (that's how we compute the matrix for any linear transformation) this shows that $A\vec{v}_1 = \vec{e}_1$ and $A\vec{v}_2 = \vec{e}_2$.

(d) Part (c) tells us that to find the first column of B, we just need to find the vector $\vec{v}_1 = (x_1, y_1)$ which is the solution to $A\vec{v}_1 = \vec{e}_1$, and to find the second column of B we need to find the vector $\vec{v}_2 = (x_2, y_2)$ which is the solution to $A\vec{v}_2 = \vec{e}_2$.

This amounts to solving two systems of equations. We could solve them separately, doing two separate row reductions:

$$\left[\begin{array}{cc:c} 9 & 5 & 1 \\ 5 & 3 & 0 \end{array}\right] \rightsquigarrow \left[\begin{array}{cc:c} 1 & 0 & \frac{3}{2} \\ 0 & 1 & -\frac{5}{2} \end{array}\right] \text{ and } \left[\begin{array}{cc:c} 9 & 5 & 0 \\ 5 & 3 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{cc:c} 1 & 0 & -\frac{5}{2} \\ 0 & 1 & \frac{9}{2} \end{array}\right].$$

But parts (a) and (b) tell us that we get the same answer if we just do it all at once:

$$\left[\begin{array}{cc:cc} 9 & 5 & 1 & 0 \\ 5 & 3 & 0 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{cc:cc} 1 & 0 & \frac{3}{2} & -\frac{5}{2} \\ 0 & 1 & -\frac{5}{2} & \frac{9}{2} \end{array}\right].$$

And that's why row reducing $\left[A \vdots I_2\right]$ gives $\left[I_2 \vdots B\right]$, with B the inverse of A.