1. Plugging in the values $t = -1$, $t = 2$, and $t = -3$ into the polynomial $a+bt+ct^2+dt^3$, we see that the map T from $\mathbb{R}^4$ to $\mathbb{R}^3$ described is given by the formula

$$T(a,b,c,d) = (a - b + c - d,\ a + 2b + 4c + 8d,\ a - 3b + 9c - 27d).$$

(a) Here are two equally valid ways to see that this is a linear transformation:

(i) From the explicit formula, plugging the vector (a,b,c,d) into T has the same result as multiplying the vector by this matrix:

$$A = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 1 & 2 & 4 & 8 \\ 1 & -3 & 9 & -27 \end{bmatrix}$$

The function is therefore linear, since functions given by a matrix are linear. More generally, if a function is given by explicit formulas, and the formulas are "linear" (i.e., only involve expressions in the variables of degree 1, with no constant term), then the function is linear – the same argument applies: the function can always be given by a matrix.

(ii) We can directly verify the properties of linearity:

$$\begin{aligned} &T(a_1 + a_2, b_1 + b_2, c_1 + c_2, d_1 + d_2) = \\ &(a_1 + a_2 - b_1 - b_2 + c_1 + c_2 - d_1 - d_2, \\ &\qquad a_1 + a_2 + 2b_1 + 2b_2 + 4c_1 + 4c_2 + 8d_1 + 8d_2, \\ &\qquad\qquad a_1 + a_2 - 3b_1 - 3b_2 + 9c_1 + 9c_2 - 27d_1 - 27d_2) \end{aligned}$$

Which is the sum of

$$T(a_1,b_1,c_1,d_1) = (a_1 - b_1 + c_1 - d_1,\ a_1 + 2b_1 + 4c_1 + 8d_1,\ a_1 - 3b_1 + 9c_1 - 27d_1)$$

and

$$T(a_2,b_2,c_2,d_2) = (a_2 - b_2 + c_2 - d_2,\ a_2 + 2b_2 + 4c_2 + 8d_2,\ a_2 - 3b_2 + 9c_2 - 27d_2)$$

Similarly, for any number k,

$$\begin{aligned} T(ka,kb,kc,kd) &= (ka - kb + kc - kd,\ ka + 2kb + 4kc + 8kd,\ ka - 3kb + 9kc - 27kd). \\ &= k\,T(a,b,c,d). \end{aligned}$$

The function T is therefore a linear transformation.

(b) The matrix for T is the matrix A given above. (Which we can find in the usual way by plugging in the vectors $\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3$, and $\vec{e}_4$). In RREF, this matrix is

$$\begin{bmatrix} 1 & -1 & 1 & -1 \\ 1 & 2 & 4 & 8 \\ 1 & -3 & 9 & -27 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

From this we see that the kernel is the span of the vector $(6, 5, -2, -1)$.

Alternatively, thinking about the map in terms of polynomials, we're looking for polynomials p of degree three such that $p(-1) = 0$, $p(2) = 0$, and $p(-3) = 0$. But saying that plugging in a value into polynomial gives zero is the same thing as saying that we've found a factor: If $p(-1) = 0$ then $(t + 1)$ must be a factor of p, if $p(2) = 0$ then $(t - 2)$ is a factor, and if $p(-3) = 0$ then $(t + 3)$ is a factor. If all three are zero, then $(t + 1)(t - 2)(t + 3)$ must be factors of p. Since we're looking for polynomials of degree 3, and this is already a polynomial of degree 3, the only possibility is multiples of this polynomial. I.e., since

$$(t + 1)(t - 2)(t + 3) = -6 - 5t + 2t^2 + t^3,$$

the kernel must be scalar multiples of $(-6, -5, 2, 1)$.

(c) This question is asking for all the polynomials p for which

$$(p(-1), p(2), p(-3)) = (-11, 19, -131),$$

i.e., in terms of the coefficients of p, we're looking for all the vectors $\vec{x}$ in $\mathbb{R}^4$ with $T(\vec{x}) = (-11, 19, -131)$. We already know the kernel of T, and we know that the vector $\vec{x}_1 = (1, 5, -4, 3)$ is sent to $(-11, 19, -131)$.

The vectors $\vec{x}$ sent to $(-11, 19, -131)$ by T are therefore all of the form $\vec{x} = \vec{x}_1 + \vec{v}$, with $\vec{v}$ in $\ker(T)$, i.e., that

$$\vec{x} = (1, 5, -4, 3) + s(6, 5, -2, -1) = (1 + 6s, 5 + 5s, -4 - 2s, 3 - s) \text{ with } s \in \mathbb{R}$$

are all the vectors with this property. Rewriting this in terms of polynomials, we see that the polynomials p which have the values we want are all of the form

$$p(t) = (1 + 6s) + (5 + 5s)t + (-4 - 2s)t^2 + (3 - s)t^3.$$

for any $s \in \mathbb{R}$.

(d) Taking the derivative of the polynomial $p_{\vec{v}} = a + bt + ct^2 + dt^3$ and plugging in $t = 4$, we see that the formula for $p'_{\vec{v}}(4)$ in terms of a, b, c, and d is

$$p'_{\vec{v}}(4) = b + 8c + 48d$$

So, the vectors $\vec{v} = (a, b, c, d)$ such that $p'_{\vec{v}}(4) = 0$ are exactly the vectors with

$$b + 8c + 48d = 0$$

How can we see that this is a subspace? The fastest way is to say that it is the kernel of a linear map, the linear transformation $T' : \mathbb{R}^4 \longrightarrow \mathbb{R}$ given by $T'(a, b, c, d) = b + 8c + 48d$, or equivalently , the one given by the matrix

$$\begin{bmatrix} 0 & 1 & 8 & 48 \end{bmatrix}.$$

An alternate way to show that this is a subspace is to use the formula to verify the subspace properties directly.

2. *Kernel Puzzlers*

(a) If A is an $n \times p$ matrix, and B a $p \times m$ matrix, with $\ker(A) = \operatorname{im}(B)$, then the $n \times m$ matrix AB is the zero matrix, the matrix where every entry is zero.

Why is that? Well, since multiplication of matrices represents composition of the linear tranformations, the conditions tell us that for any vector $\vec{v}$ in $\mathbb{R}^m$, $B\vec{v}$ is in the kernel of A, and so $A(B\vec{v}) = \vec{0}$. So putting each of $\vec{e}_1$, $\vec{e}_2$, $\dots$, $\vec{e}_m$ through $C = AB$ will give the zero vector. This means that the all the columns of C are zero vectors (since that's how we find the columns of the matrix), and so C is the zero matrix.

In fact, we just need that $\operatorname{im}(B)$ is contained in $\ker(A)$ for this to be true – we don't need them to be actually equal.

(b) The relation is that $\ker(C)$ is the intersection of $\ker(A)$ and $\ker(B)$.

If A is the matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{p1} & a_{p2} & \cdots & a_{pm} \end{bmatrix}$$

Then vectors in the kernel of A are all the vectors $\vec{x} = (x_1, \ldots, x_m)$ which satisfy the equations:

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1m}x_m &= 0 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2m}x_m &= 0 \\
\vdots \quad\quad \vdots \quad\quad\quad\quad\quad \vdots \quad &= \vdots \\
a_{p1}x_1 + a_{p2}x_2 + \cdots + a_{pm}x_m &= 0
\end{aligned}$$

And if B is the matrix

$$B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1m} \\ b_{21} & b_{22} & \cdots & b_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ b_{q1} & b_{q2} & \cdots & b_{qm} \end{bmatrix}$$

Then vectors in the kernel of B are all the vectors $\vec{x} = (x_1, \ldots, x_m)$ which satisfy the equations:

$$\begin{aligned}
b_{11}x_1 + b_{12}x_2 + \cdots + b_{1m}x_m &= 0 \\
b_{21}x_1 + b_{22}x_2 + \cdots + b_{2m}x_m &= 0 \\
\vdots \quad\quad \vdots \quad\quad\quad\quad\quad \vdots \quad &= \vdots \\
b_{q1}x_1 + b_{q2}x_2 + \cdots + b_{qm}x_m &= 0
\end{aligned}$$

The effect of "stacking" matrix A on top of matrix B and then asking for the kernel is just to look for vectors $\vec{x} = (x_1, \ldots, x_m)$ which satisfy all the equations at once. In other words, we're looking for exactly those vectors $\vec{x}$ which are in the kernel of A and in the kernel of B, i.e., those vectors in the intersection of $\ker(A)$ and $\ker(B)$.

(c) Yes, if $\ker(A^3) = \ker(A^2)$, then $\ker(A^4) = \ker(A^3)$.

Since matrix multiplication corresponds to composition of functions, any vector in the kernel of A will be in the kernel of AA (going through the first A already sends it to zero). Similarly, any vector in the kernel of A^2 will be in the kernel of A^3. What's surprising is that any vector in the kernel of A^3 is also in the kernel of A^2. In other words, if we start with a vector $\vec{w}$ and put it through A three times and get the zero vector, it must have already been zero after going through A twice.

Suppose we have a vector $\vec{v}$ which is in the kernel of A^4, i.e., putting it through A four times gives you zero. If we let $\vec{w} = A\vec{v}$, then this means that putting $\vec{w}$ through A^3 gives the zero vector. Since $\ker(A^3) = \ker(A^2)$, this means that putting $\vec{w}$ through A^2 also gives the zero vector, or, rewriting this in terms of $\vec{v}$ that putting $\vec{v}$ through A^3 gives the zero vector.

This shows that any vector in the kernel of A^4 is already in the kernel of A^3. Since any vector in the kernel of A^3 is automatically in the kernel of A^4, the two kernels are equal.

(d) If $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is a linear transformation, and $\vec{v}_1, \ldots, \vec{v}_k$ are linearly dependent vectors in $\mathbb{R}^n$, then the vectors $T(\vec{v}_1), \ldots, T(\vec{v}_k)$ must be linearly dependent too.

Since $\vec{v}_1, \ldots, \vec{v}_k$ are linearly dependent, there must be numbers $c_1, \ldots, c_k$, at least one of which is nonzero, with

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k = \vec{0}.$$

But now, applying the linear transformation T to both sides, we get:

$$\vec{0} = T(\vec{0}) = T(c_1\vec{v}_1 + \cdots + c_k\vec{v}_k) = c_1\,T(\vec{v}_1) + \cdots + c_k\,T(\vec{v}_k)$$

which is a linear dependence relation for $T(\vec{v}_1), \ldots, T(\vec{v}_k)$.

3. Here are two (among many) ways of seeing that $T(\vec{0}) = \vec{0}$ for any linear transformation T:

(i) Use the relation $T(c\vec{v}) = c\,T(\vec{v})$ with $c = 0$ and $\vec{v}$ any vector to get

$$T(\vec{0}) = T(0\vec{v}) = 0\,T(\vec{v}) = \vec{0}.$$

(ii) Use the relation $T(\vec{v} + \vec{w}) = T(\vec{v}) + T(\vec{w})$ with $\vec{w} = -\vec{v}$ to get

$$T(\vec{0}) = T(\vec{v} - \vec{v}) = T(\vec{v}) - T(\vec{v}) = \vec{0}.$$

Of course, just because $T(\vec{0}) = 0$ doesn't mean that T *has* to be linear (for example, $T : \mathbb{R}^3 \longrightarrow \mathbb{R}$ given by $T(x, y, z) = x^2 + y^2 + z^2$ isn't linear), but if $T(\vec{0}) \neq \vec{0}$ (like $T(x, y, z) = x + y - z + 2$), then we can be sure that T *isn't* a linear transformation.

4.

(a) The image is spanned by $(1,1,1)$, and $(1,2,3)$, and this is the smallest number of vectors we can use. The kernel is $\{\vec{0}\}$.

(b) The image is spanned by $(1,1,1)$, $(1,2,3)$, and $(1,5,7)$, and we need them all. One way of seeing this is to put the matrix into RREF. If we do this we get I_3, and so we see that all column vectors are linearly independent. This also means that the kernel is just the zero vector again.

(c) The image is spanned by $(1,4)$ and $(2,5)$. We don't need $(3,5)$ since $(3,5) = -\frac{5}{3}(1,4) + \frac{7}{3}(2,5)$. The kernel is spanned by $(5,-7,3)$.

(d) The image is spanned by $(1,m)$ (it's the line of slope m!), and the kernel by $(m,-1)$ (the line perpendicular to the line of slope m).

5.

(a) Row reducing, we get $\begin{bmatrix} 1 & -3 \\ -2 & 6 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix}$, from which we see that the kernel is spanned by $(3,1)$.

(b) Finding the matrix of T in the usual way (seeing where $\vec{e}_1, \ldots, \vec{e}_4$ go), we get

$$\begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & 0 & -3 \\ -2 & 0 & 6 & 0 \\ 0 & -2 & 0 & 6 \end{bmatrix}$$

which has RREF:

$$\begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

From the RREF we see that the kernel is spanned by $(3,0,1,0)$ and $(0,3,0,1)$.

(c) A general vector $\vec{v}$ in the kernel is of the form $(3t_1, 3t_2, t_1, t_2)$, corresponding to the matrix

$$M_{\vec{v}} = \begin{bmatrix} 3t_1 & 3t_2 \\ t_1 & t_2 \end{bmatrix}$$

The image of this matrix is spanned by $(3,1)$.

(d) To say that the vector $\vec{v}$ is in the kernel of T is the same as saying that the product $AM_{\vec{v}}$ is the zero matrix. (The zero 2×2 matrix what corresponds to $\vec{0} = (0,0,0,0)$ under the rule.)

What kind of condition guarantees that the product $AM_{\vec{v}}$ is the zero matrix? By question 2(a), what we need is that $\mathrm{im}(M_{\vec{v}})$ is contained in $\ker(A)$. Since $\ker(A)$ is only one dimensional, if $M_{\vec{v}}$ isn't the zero matrix we must have $\mathrm{im}(M_{\vec{v}}) = \ker(A)$, which explains the equality in parts (a) and (c).