The numbers are

$$\begin{aligned} a_{2001} &= 334,668 \\ a_{2002} &= 335,002, \\ a_{2003} &= 335,336, \\ a_{2004} &= 335,671, \\ a_{2005} &= 336,005, \quad \text{and} \\ a_{2006} &= 336,340. \end{aligned}$$

Obviously the numbers are too big to compute by writing out all the possibilities. How can we get a hold of the numbers a_n without having to list all the possible sums each time?

What we need is an organized way to count them. Following the way that we counted the number of ways to write n as a sum using only 1's and 2's, let's try this:

> Let's split the ways to write n as a sum into two categories: Those sums that involve at least one 3, and those sums that involve no 3's at all.

The sums of the second kind we already know how to count: they're ways to write n as a sum using only 1 and 2, and we know that the formula for the number of ways to do that is $\lfloor \frac{n}{2} \rfloor + 1$, where $\lfloor x \rfloor$ means "round down the number x to the nearest integer".

But the sums of the first kind are easy to count too – each of those sums has at least one 3, and if we take out one of the 3's, we get a sum adding up to $n-3$. Conversely, if we take any sum adding up to $n-3$ and add 3, we get a sum involving at least one 3, and adding up to n. In other words, the number of sums of the first kind is exactly a_{n-3}.

This gives us the recursive equation:

$$a_n = a_{n-3} + \left\lfloor \frac{n}{2} \right\rfloor + 1.$$

Now it seems like we do have a way to compute a_n for any n: the recursive equation above let's us work backwards from any n down to small numbers, where we know the answers, so the formula above will let us solve the problem.

Even though it will let us solve the problem, the formula above doesn't seem the easiest to work with (it's the "rounding down" that seems hard to get a hold of). Maybe we can do a little better.

What happens if we apply the recursion relation twice? That is apply it first to a_n, and then to a_{n-3}? That is, since (by the recursion equation)

$$a_{n-3} = a_{n-6} + \left\lfloor \frac{n-3}{2} \right\rfloor + 1,$$

if we combine that with the equation for a_n, we get

$$a_n = a_{n-3} + \left\lfloor \frac{n}{2} \right\rfloor + 1 = a_{n-6} + \left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{n-3}{2} \right\rfloor + 2.$$

Did that help at all? It seems like things only got *more* complicated.

Well, in fact it did help. The problem with $\lfloor \frac{n}{2} \rfloor$ is that it behaves differently depending on whether n is even or odd. If n is even, then $\lfloor \frac{n}{2} \rfloor = n/2$, which is an integer. If n is an odd number, then $\lfloor \frac{n}{2} \rfloor = n/2 - \frac{1}{2}$, which is again an integer. The good thing about the formula above is that if n is even, then $n-3$ is odd, and if n is odd, then $n-3$ is even. In other words, no matter if n is odd or even, each of the possibilities above occurs exactly once for n and $n-3$, and so we have

$$\left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{n-3}{2} \right\rfloor = \frac{n}{2} + \frac{n-3}{2} - \frac{1}{2} = \frac{2n-4}{2} = n-2.$$

Taking into account the $+2$ in the formula for a_n above, that means that we have

$$a_n = a_{n-6} + n,$$

and that *does* seem like a much easier formula!

At this point, we could start with $n = 2001$ and use the recursive formula over and over to compute a_{2001}, then do the same for $n = 2002$, etc. That still seems like a bit too much work to do – just to compute a_{2001} we'd have to use the recursive formula 333 times, and that seems a bit much.

Why 333 times? Because $2001 = 333 \cdot 6 + 3$, and so since each recursive step brings n down by 6, we're going to have to apply it that many times until we get back down to the numbers we know. In the case of 2001, we'd get all the way down to $a_3 = 3$.

But, if we just write out what we'd get by doing all those steps, we see that we can compute the sum directly:

$$a_{2001} = (333 \cdot 6 + 3) + (332 \cdot 6 + 3) + (331 \cdot 6 + 3) + \cdots + (2 \cdot 6 + 3) + (1 \cdot 6 + 3) + a_3.$$

Which we can rewrite as

$$a_{2001} = 6(333 + 332 + 331 + \cdots + 3 + 2 + 1) + 333 \cdot 3 + a_3.$$

We know how to add all the numbers from 1 to 333: we have a formula for that. We also know what a_3 is (and of course, we can multiply 3 times 333), so we can compute:

$$a_{2001} = 6\left(\frac{333(333+1)}{2}\right) + 999 + 3 = 334,668.$$

We can use the same reasoning for the other n's, but we can also try (again!) to do things more generally. The previous pattern for the computation shows us that what we need to know is how to write n as a multiple of 6, plus some number r which is small (say between 0 and 5). That is, let's write $n = 6k + r$ with $0 \leq r \leq 5$, i.e., r is the remainder when we divide n by 6 (For 2001, we'd have $k = 333$, $r = 3$). Then the recursion gives us

$$a_n = (6k + r) + (6(k-1) + r) + (6(k-2) + r) + \cdots + (6 \cdot 2 + r) + (6 \cdot 1 + r) + a_r,$$

which we can again rewrite as

$$\begin{aligned} a_n &= 6(k + (k-1) + (k-2) + \cdots + 2 + 1) + k \cdot r + a_r = 6\left(\frac{k(k+1)}{2}\right) + k \cdot r + a_r \\ &= 3k(k+1) + k \cdot r + a_r \end{aligned}$$

Here's a table where we use this formula to compute all the a_n's

n	k	r	a_r	$a_n = 3k(k+1) + k \cdot r + a_r$
2001	333	3	$a_3 = 3$	$3 \cdot 333(334) + 3 \cdot 333 + 3 = 334,668$
2002	333	4	$a_4 = 4$	$3 \cdot 333(334) + 4 \cdot 333 + 4 = 335,002$
2003	333	5	$a_5 = 5$	$3 \cdot 333(334) + 5 \cdot 333 + 5 = 335,336$
2004	334	0	$a_0 = 1$	$3 \cdot 334(335) + 0 \cdot 333 + 1 = 335,671$
2005	334	1	$a_1 = 1$	$3 \cdot 334(335) + 1 \cdot 333 + 1 = 336,005$
2006	334	2	$a_2 = 2$	$3 \cdot 334(335) + 2 \cdot 333 + 2 = 336,340$

Well, that gives us the answer, and it's certainly getting easier and easier to calculate these numbers. Could we do any better than this?

In fact, we can, but now it gets a bit trickier. I'd like to figure out a formula for a_n where I don't have to break n down into the form $n = 6k + r$. I guess I'm wondering if there is a formula just in terms of n, and if I can take the above formula and somehow wrestle it into that form.

Let's suppose that I know what r is (for our particular n). Then $k = (n - r)/6$. If we put this into the formula, we get

$$a_n = 3k(k+1) = k \cdot r + a_r = 3\left(\frac{n-r}{6}\right)\left(\frac{n-r}{6}+1\right) + \left(\frac{n-r}{6}\right) \cdot r + a_r$$

Expanding this, we get

$$a_n = \left(\frac{n^2}{12} + \frac{n}{2}\right) - \left(\frac{r^2}{12} + \frac{r}{2}\right) + a_r.$$

Which is getting better and better. Now we only need to know the remainder r when we divide n by 6 and the above formula will give us the answer. But we can even remove the part of the formula involving r: If we notice, for $r = 0, 1, 2, 3, 4, 5$, the values of $-\frac{r^2}{12} - \frac{r}{2} + a_r$ are:

r	$a_r - \frac{r^2}{12} - \frac{r}{2}$
0	1
1	$\frac{5}{12}$
2	$\frac{2}{3}$
3	$\frac{3}{4}$
4	$\frac{2}{3}$
5	$\frac{5}{12}$

and that means that $\left(\frac{n^2}{12} + \frac{n}{2}\right) - \left(\frac{r^2}{12} + \frac{r}{2}\right) + a_r$ must be equal to $\left\lfloor \frac{n^2}{12} + \frac{n}{2} + 1 \right\rfloor$, the integer we get by rounding down $\frac{n^2}{12} + \frac{n}{2} + 1$ whenever $1 \le r \le 5$, and equal to $\frac{n^2}{12} + \frac{n}{2} + 1$ when $r = 0$.

Since the round down of an integer is just that same integer, this means that we in fact have

$$a_n = \left\lfloor \frac{n^2}{12} + \frac{n}{2} + 1 \right\rfloor$$

no matter what n is, which is perhaps the most efficient formula, although not necessarily the prettiest.