DUE DATE: NOV. 24, 2004

1. Let's finally prove that "dimension" is a well defined notion.

(a) If V is a subspace of $\mathbb{R}^n$, and $v_1, v_2, \ldots, v_q$ vectors in V which are linearly independent, and $w_1, w_2, \ldots w_p$ vectors in V which span V, show that $q \leq p$.

HINT: Let A be the $n \times q$ matrix whose columns are $v_1, \ldots, v_q$, and B be the $n \times p$ matrix whose columns are the vectors $w_1, \ldots, w_p$. Since the w's span V, and since the v's are in V, explain why this means that there must be a matrix M with $A = BM$.

If $q > p$, explain why there is a nonzero vector in the kernel of BM, and therefore a nonzero vector in the kernel of A. Explain why this contradicts the fact that the v's are linearly independent, and that therefore we must have $q \leq p$.

(b) If $v_1, v_2, \ldots, v_q$ are a basis of V, and if $w_1, \ldots, w_p$ are another basis of V, show that $p = q$, i.e., show that any two bases of V have the same number of vectors, and hence the number $\dim(V)$ is well defined.

2. Summarize the argument that every subspace has a basis. Explain how we actually showed something more precise: If V is a subspace of $\mathbb{R}^n$, and $v_1, \ldots, v_r$ vectors in V which are linearly independent, then we can extend $v_1, \ldots, v_r$ to a basis for V, i.e., there is a basis for V for which $v_1, \ldots, v_r$ make up the first r vectors.

Use this more precise form to show that if W and V are both subspaces of $\mathbb{R}^n$, and if $W \subseteq V$ then $\dim(W) \leq \dim(V)$. Suggested argument: Pick a basis $w_1, \ldots, w_r$ of W. By definition of basis these are all linearly independent. But since $W \subseteq V$, these are also vectors in V. Now explain how the more precise form above shows that $\dim(W) \leq \dim(V)$. (we know that dim makes sense and is the number of vectors in any basis, thanks to question 1 above).

3. The purpose of this question is to emphasize that, even if something we're working with doesn't look like a "number", as long as it obeys all the rules that numbers obey, we can treat it and manipulate it as if it were a number. In particular, I'd like to try this with the 2×2 matrices of the special form

$$\begin{bmatrix} u & v \\ -v & u \end{bmatrix}.$$

We didn't check that they obey all the rules we'd expect numbers to obey, but let's check the two least obvious ones now:

(a) Show that two matrices of this type always commute. I.e., if A and B are two matrices of this type, show that $AB = BA$.

(b) Show that every nonzero matrix of this type has an inverse of this type. To be more concrete, if

$$A = \begin{bmatrix} u & v \\ -v & u \end{bmatrix},$$

with either u or v nonzero, show that the matrix

$$\begin{bmatrix} \frac{u}{u^2+v^2} & \frac{-v}{u^2+v^2} \\ \frac{v}{u^2+v^2} & \frac{u}{u^2+v^2} \end{bmatrix}$$

is the inverse of A.

(c) Just in case it's useful later, if $D = \begin{bmatrix} 3 & -5 \\ 5 & 3 \end{bmatrix}$, compute D^2.

(d) Find all solutions X (in 2×2 matrices of the above type) to the equation

$$\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} X^2 + \begin{bmatrix} 1 & -11 \\ 11 & 1 \end{bmatrix} X + \begin{bmatrix} -12 & 14 \\ -14 & -12 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

No linear algebra is needed to solve this equation. Check your answers by substituting your solutions back into the equation.

4. Let V be the set of infinite sequences $(x_1, x_2, x_3, \ldots\ldots)$ with the definition of addition and multiplication by scalars from class. Decide if the following two subsets of V are actually subspaces of V, and explain why or why not.

(a) Sequences of the form $(a, a+k, a+2k, a+3k, \ldots)$ for some a and k. i.e., $(1, 3, 5, 7, \ldots)$ and $(-4, -5, -6, -7, \ldots)$ are in the subset, since the first one is of the above type with $a = 1$, $k = 2$, and the second is of the above type with $a = -4$, $k = -1$. However, anything beginning with $(1, 3, 4, 7, \ldots)$ could not be in the subset, since there is no a and k which matches that pattern.

(b) Sequences of the form $(a, ar, ar^2, ar^3, \ldots)$ for some a and r (similar interpretation as in part (a)).

5. If B is a 4×4 diagonal matrix, check that the set of 4×4 matrices A which commute with B (i.e., those matrices A so that $AB = BA$) form a subspace of the 4×4 matrices.

What are the possibilities for the dimensions of these subspaces? (The answer depends on the diagonal matrix B, i.e., there is more than one possibility. For instance, if $B = I_4$, the 4×4 identity matrix, then all the 4×4 matrices commute with B, so the answer is 16. If B is a different diagonal matrix, the dimension might be different. The question is to find all the possible dimensions, and explain how to figure out the dimension just by looking at B.)