DUE DATE: DEC. 1, 2004

1. Define a sequence of integers by $a_0 = 3$, $a_1 = 6$, $a_2 = 14$, and for $n \geq 3$,

$$a_{n+3} = 6a_{n+2} - 11a_{n+1} + 6a_n.$$

For instance, $a_3 = 36$, since $6a_2 - 11a_1 + 6a_0 = 6(14) - 11(6) + 6(3) = 36$, and $a_4 = 98$ since $6a_3 - 11a_2 + 6a_1 = 6(36) - 11(14) + 6(6) = 98$.

Find (and prove) a formula for $a_n$ for all $n \geq 0$.

SUGGESTED OUTLINE: Solve this in the same way we found a formula for the Fibonacci numbers. Look at the set $V$ of infinite sequences $(x_0, x_1, x_2, x_3, \ldots\ldots)$ which satisfy the recursion conditions $x_{n+3} = 6x_{n+2} - 11x_{n+1} + 6x_n$ for all $n \geq 0$. The sequence $(a_0, a_1, a_2, \ldots)$ is part of this set.

Briefly indicate why $V$ is a subspace of the vector space of all possible sequences, and explain what its dimension is and why.

Look for elements of $V$ of the form

$$(1, \alpha, \alpha^2, \alpha^3, \alpha^4, \alpha^5, \ldots\ldots).$$

Can you find enough elements of this type to form a basis of $V$?

If so, you can write the $a$ sequence (which is an element of $V$) as a linear combination of these elements, and then find a formula for $a_n$.

2. *Polynomial Interpolation:*

Let $P_5$ be the vector space of polynomials of degree $\leq 5$.

(a) What is the dimension of $P_5$?

Suppose that $x_1$, $x_1$, $\ldots$, $x_6$ are any six distinct real numbers, and $y_1$, $y_2$, $\ldots$, $y_6$ any real numbers at all. I'd like to prove that there is a unique polynomial $p$ in $P_5$ with $p(x_1) = y_1$, $p(x_2) = y_2$, $\ldots$, $p(x_6) = y_6$. In other words, that given any six different $x$-values, and any six $y$-values, there is a unique polynomial of degree $\leq 5$ with height $y_i$ over $x_i$, $i = 1, 2, \ldots, 6$.

Define a map $T : P_5 \longrightarrow \mathbb{R}^6$ by

$$T(p) = (p(x_1), p(x_2), \ldots, p(x_6))$$

i.e., by plugging in $x_1$, through $x_6$.

(b) Explain why $T$ is a linear map from $P_5$ to $\mathbb{R}^6$.

(c) Explain why the statement "there is a unique $p$ in $P_5$ with $p(x_i) = y_i$ for $i = 1, \ldots, 6$" is the same as saying that $\ker(T) = \vec{0}$ and $\mathrm{im}(T) = \mathbb{R}^6$.

(d) Explain why the above is the same as saying that $\ker(T) = \vec{0}$.

(e) What does it mean in terms of $x_1, \ldots, x_6$ for a polynomial $p$ to be in $\ker(T)$? By thinking about factoring, show that $\ker(T) = \vec{0}$, the zero polynomial.

3.

(a) If $T_1 : V_1 \longrightarrow V_2$ and $T_2 : V_1 \longrightarrow V_2$ are linear transformations between the same two vector spaces $V_1$ and $V_2$, show that $T_1 + T_2$, defined by

$$(T_1 + T_2)(f) = T_1(f) + T_2(f)$$

for all $f$ in $V_1$ is also a linear transformation from $V_1$ to $V_2$.

(b) If $T : V_1 \longrightarrow V_2$ is a linear transformation from $V_1$ to $V_2$, and $c$ any number, show that $cT$, defined by

$$(cT)(f) = c(T(f))$$

for any $f$ in $V_1$ is a linear transformation from $V_1$ to $V_2$.

(c) What parts (a) and (b) show is that, given any two vector spaces $V_1$ and $V_2$, the set of possible linear transformations from $V_1$ to $V_2$ is also a vector space. If $V_1$ is $\mathbb{R}^3$, and $V_2$ is $\mathbb{R}^2$, what is the dimension of this vector space, the vector space of all linear maps from $\mathbb{R}^3$ to $\mathbb{R}^2$?

4. If $p(x)$ is a polynomial of degree $d$, say $p(x) = c_d x^d + c_{d-1}x^{d-1} + \cdots + c_1 x + c_0$, and $A$ is any $n \times n$ matrix (i.e., a square matrix) then let's interpret $p(A)$ as the $n \times n$ matrix

$$c_d A^d + c_{d-1}A^{d-1} + \cdots + c_1 A + c_0 I_n.$$

i.e., since $A$ to any power is still an $n \times n$ matrix, it makes sense to add all these $n \times n$ matrices and get a new matrix. The only thing in the expression which wasn't automatically an $n \times n$ matrix was the "constant term", which is why we added $I_n$, the $n \times n$ identity matrix.

(a) Show that for any square matrix $A$, there is some nonzero polynomial $p(x)$ of degree $\leq n^2$ with $p(A) = 0$ (here 0 means the zero matrix).

(b) If $A = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$, find explicitly such a polynomial $p(x)$ of degree $\leq 2$.

(Later in the course we'll see that we can always find such a polynomial of degree $\leq n$, and that this polynomial says important things about $A$.)