DUE DATE: OCT. 20, 2004

1. Let's consider a vector $\vec{v} = (a, b, c, d)$ in $\mathbb{R}^4$ to be the coefficients of a degree three polynomial:

$$p_{\vec{v}}(t) = a + b\,t + c\,t^2 + d\,t^3.$$

Here the subscript on p is to remind us that the polynomial depends on the vector $\vec{v}$.

For any values of t, we can plug in that value into $p_{\vec{v}}$ and get a number. For instance, if $t = 1$, and $\vec{v} = (1, -3, 4, 2)$, then $p_{(1,-3,4,2)}(1) = 1 - 3 \cdot 1 + 4 \cdot 1^2 + 2 \cdot 1^3 = 4$.

Arbitrarily picking the three t values $t = -1$, $t = 2$, and $t = -3$, we get a map $\mathbb{R}^4$ to $\mathbb{R}^3$ by $\vec{v} \mapsto (p_{\vec{v}}(-1),\, p_{\vec{v}}(2),\, p_{\vec{v}}(-3))$.

(a) Show that this is a linear transformation from $\mathbb{R}^4$ to $\mathbb{R}^3$.

(b) Find the kernel of this linear transformation.

(c) The polynomial $q(t) = 1 + 5\,t - 4\,t^2 + 3\,t^3$ has the value -11 at $t = -1$, 19 when $t = 2$, and -131 when $t = -3$. Find all the other polynomials of degree ≤ 3 that have the same values at those points.

(d) Look at all the vectors $\vec{v}$ in $\mathbb{R}^4$ so that $p'_{\vec{v}}(4) = 0$. Does this form a subspace of $\mathbb{R}^4$? Explain why or why not. (p' means the derivative of p.)

2. *Kernel puzzlers*

(a) If A is an $n \times p$ matrix, and B is a $p \times m$ matrix, with $\ker(A) = \operatorname{im}(B)$, what can you say about the product AB?

(b) if A is a $p \times m$ matrix, and B a $q \times m$ matrix, and we make a $(p + q) \times m$ matrix C by "stacking" A on top of B:

$$C = \begin{bmatrix} A \\ B \end{bmatrix},$$

what is the relation between $\ker(A)$, $\ker(B)$, and $\ker(C)$?

(c) Consider a square matrix A with $\ker(A^2) = \ker(A^3)$. Is $\ker(A^3) = \ker(A^4)$? Justify your answer. (Note, A^2 is shorthand for the matrix product AA, A^3 for AAA, etc.)

(d) Consider a linear transformation $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$. If $\vec{v}_1, \ldots, \vec{v}_k$ are linearly dependent vectors in $\mathbb{R}^n$, are the vectors $T(\vec{v}_1), \ldots, T(\vec{v}_k)$ necessarily linearly dependent in $\mathbb{R}^m$? If so, why?

3. By the zero vector $\vec{0}$ in $\mathbb{R}^n$ we mean the vector where all the entries are zero. If $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is a linear transformation, show that $T(\vec{0}) = \vec{0}$. How many ways can you think of showing that this has to happen?
(NOTE 1: The first $\vec{0}$ is a vector in $\mathbb{R}^n$, the second a vector in $\mathbb{R}^m$. NOTE 2: This should be easy, and this is perhaps the first thing you should check to see if a function could be linear.)

4. Find the image and kernel of the following matrices. For each of them, use the smallest number of vectors possible spanning the image and kernel to describe them.

(a) $\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 5 \\ 1 & 3 & 7 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 5 \end{bmatrix}$

(d) The matrix for the linear transformation T_m (projection onto a line of slope m) from Homework 3, Question 4(a).

5. Let's consider a vector $\vec{v} = (a, b, c, d)$ in $\mathbb{R}^4$ as representing a 2×2 matrix:

$$M_{\vec{v}} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Again, we use the notation $M_{\vec{v}}$ to remind us that the matrix depends on $\vec{v}$.

Let A be the matrix $A = \begin{bmatrix} 1 & -3 \\ -2 & 6 \end{bmatrix}$.

For any vector $\vec{v}$ in $\mathbb{R}^4$, the product $AM_{\vec{v}}$ is again a 2×2 matrix, which we can then consider again as a vector in $\mathbb{R}^4$. This gives a map $T : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$.

For instance, the vector $\vec{v} = (1, -3, 4, 2)$ corresponds to the matrix $M_{\vec{v}} = \begin{bmatrix} 1 & -3 \\ -2 & 6 \end{bmatrix}$, and the product is $AM_{\vec{v}} = \begin{bmatrix} -11 & -6 \\ 22 & 12 \end{bmatrix}$ corresponding to the vector $(-11, -6, 22, 12)$.

Therefore $T(1, -3, 4, 2) = (-11, -6, 22, 12)$.

(a) Find the kernel of A (a subspace of $\mathbb{R}^2$).

(b) Find the kernel of T (a subspace of $\mathbb{R}^4$).

(c) If $\vec{v}$ is any vector (except the zero vector) in the kernel of T, describe the image of $M_{\vec{v}}$.

(d) Can you explain the connection between (a) and (c)? Is there a connection between (a) and (c)?