DUE DATE: OCT. 27, 2004

1. The linear transformation $T : \mathbb{R}^4 \longrightarrow \mathbb{R}^3$ is given by the matrix

$$A = \begin{bmatrix} 2 & 0 & 6 & 1 \\ 3 & 1 & 11 & 0 \\ -3 & 0 & -9 & 1 \end{bmatrix}, \text{ which has RREF } \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

which is something that you don't have to prove.

(a) Find a basis for the image of $T$.

(b) The vector $\vec{b} = \begin{bmatrix} 5 \\ 5 \\ 0 \end{bmatrix}$ is in the image. Find the linear combination of basis vectors from (a) which gives $\vec{b}$. Use this to find a vector $\vec{x}$ in $\mathbb{R}^4$ with $T(\vec{x}) = \vec{b}$.

(c) Find a basis for the kernel of $T$.

(d) Write down all the solutions to $T(\vec{x}) = \vec{b}$, with $\vec{b}$ the vector from part (b). You shouldn't have to do any complicated calculations to figure this out.

(e) Find all solutions to the system of equations

$$\begin{array}{rcccccccc} 2x & + & & + & 6z & + & w & = & 5 \\ 3x & + & y & + & 11z & & & = & 5 \\ -3x & & & + & -9z & + & w & = & 0 \end{array}$$

using the "old" method – the way of solving equations that we learned in the first weeks of class.

(f) Explain the connection between (d) and (e).

2. If $A$ is an $n \times m$ matrix, and $B$ an *invertible* $n \times n$ matrix,

(a) What is the relation between $\ker(A)$ and $\ker(BA)$? How do you know?

(b) What is the relation between the dimension of $\text{im}(A)$ and the dimension of $\text{im}(BA)$? Explain your argument.

(c) What is the relation between $\text{rank}(A)$ and $\text{rank}(BA)$? Explain your argument.

3. Summary 3.3.9, on page 133, gives some equivalent characterizations of an invertible $n \times n$ matrix $A$. Here are some of them:

(i) $A$ is invertible

$\vdots$

(vi) $\ker(A) = \vec{0}$.

(vii) The column vectors of $A$ form a basis of $\mathbb{R}^n$.

(viii) The column vectors of $A$ span $\mathbb{R}^n$

(ix) The column vectors of $A$ are linearly independent.

I'd like to understand why all those things are the same. For an $n \times n$ matrix $A$,

(a) Explain why (vi) and (ix) are the same thing. (This involves thinking about what vectors in the kernel tell you about the column vectors).

(b) Explain why (vi) and (viii) are the same thing. (The rank-nullity theorem is a good thing to think about here).

(c) Explain why (vii) is the same as (viii) and (ix) together.

(d) Explain why (i) is the same as (vi) and (viii) together. (The definition of what it means for a transformation to be invertible might help).

(e) If you know that any one of these properties is true, do the arguments above show you that all the others have to be true too?