DUE DATE: NOV. 3, 2004

1. Suppose that $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is a linear transformation.

   (a) If $\vec{x}$ is a vector in $\mathbb{R}^n$, and $\vec{v}$ a vector in the kernel of T, explain why $T(\vec{x}) = T(\vec{x}+\vec{v})$.

   (b) Conversely, if $\vec{x}_1$ and $\vec{x}_2$ are vectors in $\mathbb{R}^n$, and if $T(\vec{x}_1) = T(\vec{x}_2)$, explain why there is a vector $\vec{v}$ in the kernel of T with $\vec{x}_2 = \vec{x}_1 + \vec{v}$. (HINT: What does T do to $\vec{x}_2 - \vec{x}_1$?)

   (c) If $\vec{b}$ is a vector in $\mathbb{R}^m$, and $\vec{x}_1$ a vector in $\mathbb{R}^n$ with $T(\vec{x}) = \vec{b}$, explain why the solutions to the equation $T(\vec{x}) = \vec{b}$ are all of the form $\vec{x} = \vec{x}_1 + \vec{v}$, with $\vec{v}$ in $\ker(T)$.

2. Suppose that T is the linear transformation $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ given by

$$T(x, y) = (6x - 3y,\ 2x - y,\ 4x - 2y).$$

Notice that the image of T is the line in $\mathbb{R}^3$ spanned by $(3, 1, 2)$.

   (a) Find and describe the kernel of T.

   (b) Find all the points in $\mathbb{R}^2$ that map to $(3, 1, 2)$ under T.

   (c) Find all the points in $\mathbb{R}^2$ that map to $(-6, -2, -4)$ under T.

   (d) Sketch (in $\mathbb{R}^2$), the answers from parts (a), (b), and (c).

   (e) Explain how your sketch relates to the computations in question (1) above.

3. Suppose that A is a 3×2 matrix, and B is a 2×3 matrix.

   (a) Explain why the 3×3 matrix AB can never be invertible. (HINT: what are the possibilities for the dimension of the image of AB?)

   (b) Find an example where the 2×2 matrix BA *is* invertible.

   (c) If BA is invertible, what must be the dimension of $\ker(A)$, and what must be the dimension of $\ker(B)$? Explain why.

4. I don't know if you've seen these before, but there are identities among sin and cos concerning angle addition. That is, if α and θ are two angles, there is a way to write $\sin(\alpha + \theta)$ in terms of sin's and cos's of α and θ, and the same thing for $\cos(\alpha + \theta)$. There are purely geometric proofs of these identies using triangles, but here's a linear algebra proof:

Let T_α be the linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ which is rotation counterclockwise by α, and T_θ the counterclockwise rotation by θ.

(a) Explain what the linear transformation $T_\alpha \circ T_\theta$ does to $\mathbb{R}^2$.

(b) Compute the matrix for $T_\alpha \circ T_\theta$ by multiplying the matrices for T_α and T_θ.

(c) On the other hand, from the description in part (a), you can write down directly the matrix for $T_\alpha \circ T_\theta$. What is that matrix?

(d) Since the matrices from parts (b) and (c) are equal (they describe the same linear transformation) what identities among sin and cos must be true?

(e) Using a similar idea, find formulas for $\sin(3\theta)$ and $\cos(3\theta)$ in terms of $\sin(\theta)$ and $\cos(\theta)$.

5. Here are three short problems on matrix squaring and images:

(a) Find a 2×2 matrix A with $\text{im}(A) = \ker(A)$ as subspaces of $\mathbb{R}^2$.

(b) If A is a 10×10 matrix with $A^2 = 0$ (the zero matrix), show that $\text{rank}(A) \leq 5$.

(c) If A is an $n \times n$ matrix with $A^2 = A$, show that the only vector that $\ker(A)$ and $\text{im}(A)$ have in common is the zero vector.