1. Suppose that $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is a function, and we know that this function obeys these two rules:

A: $T(\vec{v} + \vec{w}) = T(\vec{v}) + T(\vec{w})$ for any two vectors $\vec{v}$, $\vec{w}$ in $\mathbb{R}^n$.

B: $T(c\vec{v}) = c\,T(\vec{v})$ for any vector $\vec{v}$ in $\mathbb{R}^n$, and any number c.

I'd like to know if this means that T also has to obey this rule:

C: $T(c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_r\vec{v}_r) = c_1\,T(\vec{v}_1) + c_2\,T(\vec{v}_2) + \cdots + c_r\,T(\vec{v}_r)$, for any vectors $\vec{v}_1, \ldots, \vec{v}_r$ in $\mathbb{R}^n$, and any numbers $c_1, \ldots, c_r$.

Prove that if T satisfies A and B together it must also satisfy C. Prove it the other way around too: if T obeys rule C, then A and B also have to be true.

Note that rule A only says that you can move the sum of two vectors through the brackets – it doesn't say that you can move three – i.e., rule A does not directly state that $T(\vec{u}+\vec{v}+\vec{w}) = T(\vec{u})+T(\vec{v})+T(\vec{w})$, so if you want to use anything like that you'll have to explain how it follows from rule A just as it is.

This problem is not meant to be hard, nor the solution difficult. It's purpose is just to give you a chance to think about these properties again, and to think about what constitutes a mathematical argument.

2. Prove or disprove: If A is an $n \times m$ matrix, and B an $m \times p$ matrix, then

$$\mathrm{RREF}(AB) = \mathrm{RREF}(A) \cdot \mathrm{RREF}(B).$$

3. Inductorama!

(a) Prove (by induction or any other means) the formula

$$a + ar + ar^2 + ar^3 + \cdots + ar^{m-1} + ar^m = \frac{a(r^{m+1} - 1)}{r - 1},$$

valid for any number a and any $r \neq 1$.

(b) If A is the matrix $A = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}$, find and prove a formula for the entries of A^n, $n \geq 0$. Does this formula work for all n? (Something like A^{-5} should be interpreted as $(A^{-1})^5$, i.e., the inverse of A to the fifth power.)

4. Suppose that A is an $n \times n$ matrix, and we know that for some $m \geq 0$, A^m is the zero matrix. Prove that A^n is the zero matrix. (If $m \leq n$ that's not a problem. The issue is if $m > n$ to show that the smaller power A^n is already zero.)

Here are two sketches of possible arguments. Feel free to fill in the details of either one of them, or to come up with your own.

(I) Consider the sequence of dimensions $\dim(\ker(A))$, $\dim(\ker(A^2))$, $\dim(\ker(A^3))$, ... We know that eventually this has to arrive at n, since the kernel of A^m, the zero matrix, is all of $\mathbb{R}^n$, which is n dimensional.

Explain why these numbers are increasing:

$$\dim(\ker(A)) \leq \dim(\ker(A^2)) \leq \dim(\ker(A^3)) \leq \cdots$$

Show that if two consecutive numbers in the sequence are equal, e.g. if $\dim(\ker(A^r)) = \dim(\ker(A^{r+1})$ for some r then all of the numbers which follow it are equal too, i.e., that $\dim(\ker(A^{r+1})) = \dim(\ker(A^{r+2}))$, $\dim(\ker(A^{r+2})) = \dim(\ker(A^{r+3}))$, etc.

Explain why this means that we must have $\dim(\ker(A^r)) = n$, and why r must be less than or equal to n.

(II) Let r be the smallest positive integer so that A^r is the zero matrix. Since A^{r-1} isn't the zero matrix, explain why there must be a vector $\vec{v}$ with $A^{r-1}\vec{v} \neq \vec{0}$.

Show that the vectors $\vec{v}$, $A\vec{v}$, $A^2\vec{v}$, $\ldots$, $A^{r-1}\vec{v}$ are all linearly independent. One way to try this is to start with a supposed linear relation:

$$c_0\vec{v} + c_1 A\vec{v} + c_2\vec{v} + \cdots + c_{r-1}A^{r-1}\vec{v} = \vec{0}$$

and multiply both sides by A^{r-1} to show that $c_0 = 0$. Then show that $c_1 = 0$, etc.

Finish by explaining why this shows that $r \leq n$.