INTERSECTION MULTIPLICITY ONE FOR CLASSICAL GROUPS

IVAN DIMITROV AND MIKE ROTH

ABSTRACT. In this note we show that when G is a classical semi-simple algebraic group, $\mathrm{B} \subset \mathrm{G}$ a Borel subgroup, and $\mathrm{X} = \mathrm{G}/\mathrm{B}$, then the structure coefficients of the Belkale-Kumar product $\odot_0$ on $\mathrm{H}^*(\mathrm{X}, \mathbf{Z})$ are all either 0 or 1.

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1. INTRODUCTION

1.1. Let G be a semi-simple algebraic group over an algebraically closed field of characteristic zero, $\mathrm{B} \subset \mathrm{G}$ a Borel subgroup, and set $\mathrm{X} = \mathrm{G}/\mathrm{B}$.

For any element w of the Weyl group $\mathcal{W}$ of G the *Schubert variety* X_w is defined by

$$\mathrm{X}_w := \overline{\mathrm{B}w\mathrm{B}/\mathrm{B}} \subseteq \mathrm{G}/\mathrm{B} = \mathrm{X}.$$

Recall that the classes of the Schubert cycles $\{[\mathrm{X}_w]\}_{w\in\mathcal{W}}$ give a basis for the cohomology ring $\mathrm{H}^*(\mathrm{X}, \mathbf{Z})$ of X. Each $[\mathrm{X}_w]$ is a cycle of complex dimension $\ell(w)$, where $\ell(w)$ is the length of w. The dual Schubert cycles $\{[\Omega_w]\}_{w\in\mathcal{W}}$, given by $\Omega_w := \mathrm{X}_{w_0 w}$, where $w_0 \in \mathcal{W}$ is the longest element, also form a basis. Each $[\Omega_w]$ is a cycle of complex codimension $\ell(w)$.

For any $w_1, w_2, w \in \mathcal{W}$ the *structure constant* $c^w_{w_1,w_2}$ is defined to be the coefficient of $[\Omega_w]$ when expressing the product $[\Omega_{w_1}] \cdot [\Omega_{w_2}]$ as a sum of basis elements, so that

$$[\Omega_{w_1}] \cdot [\Omega_{w_2}] = \sum_{w\in\mathcal{W}} c^w_{w_1,w_2}[\Omega_w].$$

In [BK] Belkale and Kumar define a new product $\odot_0$ on $\mathrm{H}^*(\mathrm{X}, \mathbf{Z})$. (More generally [BK] defines a new product on $\mathrm{H}^*(\mathrm{G}/\mathrm{P}, \mathbf{Z})$, where P is any parabolic, however this paper is only concerned with the case $\mathrm{P} = \mathrm{B}$.) Let $d^w_{w_1,w_2}$ be the structure coefficients of the Belkale-Kumar product, so that as above

$$[\Omega_{w_1}] \odot_0 [\Omega_{w_2}] = \sum_{w\in\mathcal{W}} d^w_{w_1,w_2}[\Omega_w].$$

The Belkale-Kumar constants $d^w_{w_1,w_2}$ are equal to the usual constants $c^w_{w_1,w_2}$ if the triple (w_1, w_2, w) is *Levi-movable* [BK, Definition 4], and zero otherwise. Specifically, let Δ^+ denote the set of positive roots of G, and $\Delta^- = -\Delta^+$ the negative roots. Following Kostant [Ko, Definition 5.10], for each $w \in \mathcal{W}$ we define the *inversion set* $\Phi_w := w^{-1}\Delta^- \cap \Delta^+$. Belkale and Kumar [BK, Theorem 43+Corollary 44] prove that

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(1.1.1)
$$d^w_{w_1,w_2} = \begin{cases} c^w_{w_1,w_2} & \text{if } \Phi_w = \Phi_{w_1} \sqcup \Phi_{w_2} \\ 0 & \text{otherwise,} \end{cases}$$

where $\sqcup$ denotes disjoint union. It is known that $\Phi_w = \Phi_{w_1} \sqcup \Phi_{w_2}$ implies $c^w_{w_1,w_2} \neq 0$ (see e.g., the discussion after Corollary 2.2.2). The following two statements are therefore equivalent:

(*i*) The structure constants of the Belkale-Kumar product $\odot_0$ on $\mathrm{H}^*(\mathrm{X}, \mathbf{Z})$ are all either 0 or 1.

(*ii*) $c^w_{w_1,w_2} = 1$ whenever $\Phi_w = \Phi_{w_1} \sqcup \Phi_{w_2}$.

It is useful to write (*ii*) in a more symmetric form. Since $w_0\Delta^+ = \Delta^-$, it follows easily that $\Phi_{w_0w} = \Delta^+ \setminus \Phi_w$, so that the condition $\Phi_w = \Phi_{w_1} \sqcup \Phi_{w_2}$ is equivalent to $\Delta^+ = \Phi_{w_1} \sqcup \Phi_{w_2} \sqcup \Phi_{w_0w}$. Furthermore, since the class $[\Omega_{w_0w}]$ is dual to $[\Omega_w]$ we have $c^w_{w_1,w_2} = [\Omega_{w_1}] \cup [\Omega_{w_2}] \cup [\Omega_{w_0w}]$. Setting $w_3 = w_0w$ we can therefore rephrase (*ii*) as

(*iii*) $[\Omega_{w_1}] \cup [\Omega_{w_2}] \cup [\Omega_{w_3}] = 1$ whenever $\Delta^+ = \Phi_{w_1} \sqcup \Phi_{w_2} \sqcup \Phi_{w_3}$.

This in turn is equivalent to the following similar statement with an arbitrary number of elements of $\mathcal{W}$:

(*iv*) $\bigcup_{i=1}^k [\Omega_{w_i}] = 1$ whenever $w_1, \ldots, w_k \in \mathcal{W}$ satisfy

(1.1.2)
$$\Delta^+ = \bigsqcup_{i=1}^{k} \Phi_{w_i}.$$

It is clear that (*iv*) implies (*iii*). The proof that (*iii*) implies (*iv*) requires a slightly longer argument, and we defer it to the Appendix in §3.

1.2. Main result.

The main theorem of this paper is that the equivalent conditions (*i*)–(*iv*) hold for any classical group G and for the exceptional group G_2, and hence for any semisimple group whose factors are of classical type or isomorphic to G_2. We state and prove this result in the form of (*iv*)[1].

Theorem (1.2.1) — If G is classical (or G_2) then condition (1.1.2) implies that $\bigcup_{i=1}^k [\Omega_{w_i}] = 1$.

1.3. Other equivalent statements.

By [BK, Corollary 44] (*ii*) is equivalent to

(*v*) $\prod_{\alpha \in \Phi_{w^{-1}}} \langle \rho, \alpha \rangle = \left(\prod_{\alpha \in \Phi_{w_1^{-1}}} \langle \rho, \alpha \rangle \right) \left(\prod_{\alpha \in \Phi_{w_2^{-1}}} \langle \rho, \alpha \rangle \right)$ whenever $\Phi_w = \Phi_{w_1} \sqcup \Phi_{w_2}$.

Here ρ is one-half the sum of the positive roots and $\langle \cdot, \cdot \rangle$ the Killing form.

[1]Although, by the equivalences above, it would suffice to prove only (*iii*), we have chosen to prove statement (*iv*) for arbitrary k since it seems useful to record the more general versions of some of the combinatorial statements used in the proof.

In [BK, Theorem 43] Belkale and Kumar give an isomorphism of graded rings:

$$\phi\colon\ (\mathrm{H}^*(\mathrm{X},\mathbf{C}),\odot_0)\cong\left[\mathrm{H}^*(\mathfrak{u}^+)\otimes\mathrm{H}^*(\mathfrak{u}^-)\right]^{\mathfrak{t}},$$

where $\mathrm{H}^*(\mathfrak{u}^\pm)$ denotes Lie algebra cohomology of the nilpotent algebras $\mathfrak{u}^\pm$, and $\mathfrak{t}$ the subalgebra corresponding to the maximal torus. Under this isomorphism

$$\phi\left([\Omega_w]\right)=(-1)^{\frac{p(p-1)}{2}}\left(\frac{i}{2\pi}\right)^p\left(\prod_{\alpha\in\Phi_{w^{-1}}}\langle\rho,\alpha\rangle\right)\xi^w$$

where $p=\ell(w)$, and where (roughly) $\xi^w=(\wedge_{i=1}^p y_{\beta_i})\otimes(\wedge_{i=1}^p y_{-\beta_i})$, with $\beta_1,\ldots,\beta_p$ the roots in Φ_w and each y_α an element in the subspace of weight α (see [BK, Theorem 43] for the precise normalizations used in the definition of ξ^w). The factors of $(\frac{i}{2\pi})$ are taken care by the grading of the cohomology groups, and if (*v*) and (*ii*) hold we may also ignore the factors $\prod\langle\rho,\alpha\rangle$. Thus an equivalent version of the above statements is

(*vi*) The map

$$\phi'\colon\ (\mathrm{H}^*(\mathrm{X},\mathbf{Q}),\odot_0)\longrightarrow\left[\mathrm{H}^*(\mathfrak{u}^+_{\mathbf{Q}})\otimes\mathrm{H}^*(\mathfrak{u}^-_{\mathbf{Q}})\right]^{\mathfrak{t}}$$

defined by

$$\phi'\left([\Omega_w]\right)=(-1)^{\frac{p(p-1)}{2}}\,\xi^w$$

is an isomorphism of graded rings, where, as above, $p=\ell(w)$.

1.4. An application to the Littlewood-Richardson cone. For any $k\geqslant 1$ let $\mathcal{C}(k)$ denote the (symmetric) *Littlewood-Richardson cone*, i.e., the rational cone generated by the k-tuples $(\mu_1,\ldots,\mu_k)$ of dominant weights of G such that $(\mathrm{V}_{\mu_1}\otimes\cdots\otimes\mathrm{V}_{\mu_k})^{\mathrm{G}}\neq 0$. It is known that $\mathcal{C}(k)$ is polyhedral. More precisely, [BK, Theorem, p. 187] shows that the supporting hyperplanes of $\mathcal{C}(k)$ are the hyperplanes $\mu_i=0$ and hyperplanes of the form

(1.4.1)
$$(\sum_{i=1}^k w_i^{-1}\mu_i,\varpi_{\mathrm{P}})=0,$$

where P is a maximal parabolic subgroup of G, ϖ_{P} is the fundamental weight of G corresponding to P, and $w_1,\ldots,w_k$ are elements of $\mathcal{W}$ satisfying the following conditions:

(*a*) w_i is the element of minimal length in $w_i\mathcal{W}_{\mathrm{P}}\subset\mathcal{W}/\mathcal{W}_{\mathrm{P}}$;
(*b*) $w_1,\ldots,w_k$ are Levi-movable;
(*c*) $[\Omega_{w_1}]\odot_0\cdots\odot_0[\Omega_{w_k}]=1$ in $\mathrm{H}^*(\mathrm{G}/\mathrm{P},\mathbf{Z})$.

A face of $\mathcal{C}(k)$ is called *regular* if it intersects the interior of the dominant chamber, i.e., if it contains a k-tuple $(\mu_1,\ldots,\mu_k)$ of strictly dominant weights. Ressayre [Re2, Theorem C] shows that the regular faces of $\mathcal{C}(k)$ are in bijection with k-tuples $(w_1,\ldots,w_k)$ satisfying (*a*)–(*c*) above with respect to some parabolic P (not necessarily maximal). The codimension of the face is the number of negative simple roots in P and the face is cut out by equations analogous to (1.4.1), one for each negative simple root of P. In particular, regular faces exist in codimensions $1,2,\ldots,\mathrm{rank}(\mathrm{G})$ and the regular faces of highest codimension are parameterized by k-tuples $w_1,\ldots,w_k$ with $\mathrm{P}=\mathrm{B}$.

When $\mathrm{P} = \mathrm{B}$, the minimality condition (*a*) is vacuous, the Levi-movability condition, (*b*), is (1.1.2), and by (*iv*) we have that (1.1.2) implies $\bigcup_{i=1}^{k}[\Omega_{w_i}] = 1$ and hence (see Lemma 3.1.2(*a*)) that $[\Omega_{w_1}] \odot_0 \cdots \odot_0 [\Omega_{w_k}] = 1$, i.e., that (*c*) holds.

Thus, combined with [Re2, Theorem C], Theorem 1.2.1, shows that (for those G listed in the theorem) the regular faces of $\mathcal{C}(k)$ of maximal codimension are in one-to-one correspondence with the k-tuples $(w_1, \ldots, w_k)$ of Weyl group elements satisfying (1.1.2).

1.5. The product $\odot$ for a general P. Finally we note that the corresponding versions of the statements (*i*)–(*vi*) do not hold for the Belkale-Kumar product on quotients G/P in general. For instance, when P is a maximal parabolic in type A, the Belkale-Kumar product on $\mathrm{H}^*(\mathrm{G}/\mathrm{P}, \mathbf{Z})$ is the usual cup-product, and there are many examples of Littlewood-Richardson coefficients different from 0 or 1.

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2. Intersection multiplicity one for classical groups

2.1. Approach. To prove Theorem 1.2.1 we will compute the intersection $\bigcup_{i=1}^{k}[\Omega_{w_i}]$ by two different methods. In types A, B, and C we will use a method combining Weyl group combinatorics and representation theory. In type D we will use a more geometric fibration method due to Richmond; our application of this method relies on a key combinatorial result in type D (Lemma 2.7.1).

In type A Theorem 1.2.1 has already been proved by Richmond [Ri1, Corollary 4], and in type C by Ressayre [Re1, Corollary 1, p. 96]. Both of these arguments use Richmond's fibration method. We now set up and apply the first, combinatorial, method.

2.2. Torus fixed points and Weyl group combinatorics. In the first approach we will compute the intersections by intersecting subvarieties representing these classes. The representatives will be torus stable subvarieties so it is useful to understand their torus fixed points; this is most easily described using the Bruhat order on $\mathcal{W}$.

Recall that the Bruhat order $\leqslant$ on $\mathcal{W}$ is the translation of the inclusion order on Schubert varieties to $\mathcal{W}$, i.e., we set $u \leqslant v$ if and only if $X_u \subseteq X_v$. If $w \in \mathcal{W}$, we denote the point $wB/B \in X_w \subseteq X$ by w for short. With this notation in mind, we recall that the torus fixed points of X are exactly the points w for $w \in \mathcal{W}$.

Lemma (2.2.1) — For any element w of $\mathcal{W}$ the torus fixed points of $(w_0w)^{-1}\Omega_w = (w_0w)^{-1}\mathrm{X}_{w_0w}$ are the elements of the set

$$\left\{ u \mid w \leqslant wu \right\}.$$

Proof. The torus fixed points of X_{w_0w} are the elements v such that $v \leqslant w_0w$, and hence the torus fixed points of $(w_0w)^{-1}\mathrm{X}_{w_0w}$ are the elements of the form $w^{-1}w_0v$ with $v \leqslant w_0w$.

Making the change of variables $u = w^{-1}w_0v$ (so that $v = w_0wu$), then this is the set of elements $\{u \mid w_0wu \leqslant w_0w\}$. Since $w_0wu \leqslant w_0w$ if and only if $wu \geqslant w$ (in general $x \leqslant y$ iff $w_0x \geqslant w_0y$) this proves the lemma. □

Corollary (2.2.2) — For any elements $w_1, \ldots, w_k \in \mathcal{W}$, the torus fixed points of the intersection $\bigcap_{i=1}^{k}(w_0w_i)^{-1}\Omega_{w_i}$ of the shifted Schubert varieties are the elements of the set

(2.2.3) $$\left\{ u \in \mathcal{W} \;\middle|\; w_i \leqslant w_iu \text{ for all } i = 1, \ldots, k \right\}.$$

The proof of [DR, Lemma (2.6.1)] shows that if $w_1, \ldots, w_k$ satisfy (1.1.2) then the intersection $\bigcap_{i=1}^{k}(w_0w_i)^{-1}\Omega_{w_i}$ is transverse at e, and that $\{e\}$ is an isolated component of the intersection. Since the schemes $(w_0w)^{-1}\Omega_{w_i}$ are all fixed by the torus, any component of their intersection must have a torus fixed point. Combining this with Corollary 2.2.2, to prove Theorem 1.2.1 it is therefore sufficient (assuming (1.1.2)) to show that

(2.2.4) $$\left\{ u \in \mathcal{W} \;\middle|\; w_i \leqslant w_iu \text{ for all } i = 1, \ldots, k \right\} = \left\{ e \right\}.$$

The Bruhat order on $\mathcal{W}$ admits a purely combinatorial description in the general setting of Coxeter groups. A consequence of this description is the following well-known result (see for example [Dix, Theorem 7.7.7(i), p. 267]), which we will use to demonstrate (2.2.4).

Proposition (2.2.5) — Let x, y be elements of $\mathcal{W}$ with $x \leqslant y$ in the Bruhat order. Then for any dominant weight λ the difference $x\lambda - y\lambda$ is a nonnegative sum of positive roots.

Lemma (2.2.6) — Suppose that $w_1, \ldots, w_k$ satisfy condition (1.1.2). Then

(*a*) for each root $\alpha \in \Delta^+$ there is a w_i such that $w_i\alpha$ is a negative root.

Further suppose that u is a solution to $w_i \leqslant w_iu$ for $i = 1, \ldots, k$. Then for any dominant weight λ we have :

(*b*) $\mu_\lambda := \lambda - u\lambda$ is a nonnegative sum of positive roots.
(*c*) $w_i\mu_\lambda$ is a nonnegative sum of positive roots for $i = 1, \ldots, k$.
(*d*) μ_λ is not a root or a multiple of a root.

Proof. Part (*a*) is obvious from condition (1.1.2). Part (*b*) follows from Proposition 2.2.5 since $u \geqslant e$ for any $u \in \mathcal{W}$. Part (*c*) follows from Proposition 2.2.5, the condition that $w_i \leqslant w_iu$, and the obvious identity $w_i\mu_\lambda = w_i\lambda - w_iu\lambda$. Part (*d*) is proved by combining (*a*) and (*c*). □

2.3. Proof of Theorem 1.2.1 in types A**,** B**,** C**.** Our strategy to show that (2.2.4) holds (and thus that Theorem 1.2.1 holds), is to assume that there is an element $u \neq e$ satisfying $w_i \leqslant w_iu$ for $i = 1, \ldots, k$ and then produce a dominant λ such that μ_λ violates Lemma 2.2.6(*d*). We now do this on a case-by-case basis.

Type A_n**.** Let $\epsilon_1, \ldots, \epsilon_{n+1}$ be a basis for the permutation representation of $\mathcal{W} = \mathrm{S}_{n+1}$, where as usual the positive roots are of the form $\epsilon_p - \epsilon_q$ with $p < q$. The fundamental weights are $\varpi_p := \epsilon_1 + \epsilon_2 + \cdots + \epsilon_p$ for $p = 1, \ldots, n$. Let $u \in \mathcal{W}$ be such that $w_i \leqslant w_i u$ for $i = 1, \ldots, k$.

If $u \neq e$ then let p be the smallest element of $\{1, \ldots, n\}$ such that $u\epsilon_p \neq \epsilon_p$. Since p is the smallest such element, $u\epsilon_j = \epsilon_j$ for $j < p$ and $u\epsilon_p = \epsilon_q$ with $q > p$ and hence $\mu := \varpi_p - u\varpi_p = \epsilon_p - \epsilon_q$ is a positive root, contradicting Lemma 2.2.6(*d*). Therefore $u = e$ is the only possibility.

Type B_n**.** Let $\epsilon_1, \ldots, \epsilon_n$ be the usual basis upon which $\mathcal{W}$ operates by signed permutations. The positive roots are $\epsilon_1, \ldots, \epsilon_n$ and elements of the form $\epsilon_p \pm \epsilon_q$ with $p < q$. Fundamental weights are $\varpi_p = \epsilon_1 + \cdots + \epsilon_p$ for $p = 1, \ldots n-1$ and $\varpi_n = \frac{1}{2}(\epsilon_1 + \epsilon_2 + \cdots + \epsilon_n)$. Let $u \in \mathcal{W}$ be such that $w_i \leqslant w_i u$ for $i = 1, \ldots, k$.

If $u \neq e$ then let p be the smallest element of $\{1, \ldots, n\}$ such that $u\epsilon_p \neq \epsilon_p$. If $u\epsilon_p = \pm\epsilon_q$ with $q > p$ then $\mu := \varpi_p - u\varpi_p = \epsilon_p \mp \epsilon_q$ is a positive root, contradicting Lemma 2.2.6(*d*). Therefore $p = q$ and we must have $u\epsilon_p = -\epsilon_p$. But then $\mu := \varpi_p - u\varpi_p$ is either twice a root (if $p < n$) or equal to a root (if $p = n$), again contradicting Lemma 2.2.6(*d*). Therefore $u = e$ is the only solution.

Type C_n**.** Let $\epsilon_1, \ldots, \epsilon_n$ be the usual basis upon which $\mathcal{W}$ operates by signed permutations. The positive roots are $2\epsilon_1, \ldots, 2\epsilon_n$ and elements of the form $\epsilon_p \pm \epsilon_q$ with $p < q$. Fundamental weights are $\varpi_p = \epsilon_1 + \cdots + \epsilon_p$ for $p = 1, \ldots, n$. Let $u \in \mathcal{W}$ be such that $w_i \leqslant w_i u$ for $i = 1, \ldots, k$.

The argument in this case is almost identical to that of B_n: If $u \neq e$ let p be the smallest element of $\{1, \ldots, n\}$ such that $u\epsilon_p \neq \epsilon_p$. Then $u\epsilon_p = \pm\epsilon_q$ with $q > p$ or $u\epsilon_p = -\epsilon_p$. In either case $\varpi_p - u\varpi_p$ is a root, contradicting Lemma 2.2.6(*d*), so we again have $u = e$ as the only solution. □

Of course, the results for B_n and C_n are equivalent – the natural isomorphism of Weyl groups respects the Bruhat order, and induces a bijection of inversion sets (taking roots of the form $\epsilon_p \pm \epsilon_q$ to $\epsilon_p \pm \epsilon_q$ and roots of the form ϵ_p to $2\epsilon_p$); the proof above in the C_n case was included since it is equally short.

We now turn to the general setup of Richmond's fibration method, which we will use in the proof of the theorem in type D.

2.4. Restrictions of inversion sets and fibrations. Recall that a subset $\mathrm{S} \subseteq \Delta^+$ is called *closed* if whenever $\alpha, \beta \in \mathrm{S}$ are such that $\alpha + \beta$ is a root then $\alpha + \beta \in \mathrm{S}$. A subset S is called *coclosed* if its complement S^{c} is closed. We will use the following result of Kostant (see [Ko, Proposition 5.10]) without further reference: a subset S of Δ^+ is closed and coclosed if and only if $\mathrm{S} = \Phi_w$ for some $w \in \mathrm{W}$; the element w is of course unique.

Definition (2.4.1) — Let $\mathrm{P} \supseteq \mathrm{B}$ be a parabolic subgroup, Δ_{P} the set of roots of the Levi subgroup of P, and $\mathcal{W}_{\mathrm{P}}$ the corresponding Weyl group, i.e., the group generated by the reflections in the roots contained in Δ_{P}, and set $\Delta_{\mathrm{P}}^+ = \Delta_{\mathrm{P}} \cap \Delta^+$. For any $w \in \mathcal{W}$ the set $\Phi_w \cap \Delta_{\mathrm{P}}^+$ is both closed and coclosed in Δ^+, since Φ_w is closed and coclosed in Δ^+. Therefore $\Phi_w \cap \Delta_{\mathrm{P}}^+ = \Phi_u$ for a unique $u \in \mathcal{W}$, and moreover $u \in \mathcal{W}_{\mathrm{P}}$. We define ϕ_{P} to be the (unique) map of sets $\phi_{\mathrm{P}} \colon \mathcal{W} \longrightarrow \mathcal{W}_{\mathrm{P}}$ such that

(2.4.2) $$\Phi_{\phi_{\mathrm{P}}(w)} = \Phi_w \cap \Delta_{\mathrm{P}}^+ \quad \text{for all } w \in \mathcal{W}.$$

The map ϕ_{P} has the following geometric meaning for projections of shifted Schubert varieties.

Proposition (2.4.3) — Let $\mathrm{P} \supseteq \mathrm{B}$ be a parabolic subgroup, $\mathrm{M} = \mathrm{G}/\mathrm{P}$ and $\pi\colon \mathrm{X} \longrightarrow \mathrm{M}$ the projection. Then for any $w \in \mathcal{W}$:

(*a*) $\pi(w^{-1}\mathrm{X}_w)$ has dimension $\left|\Phi_w \setminus \Phi_{\phi(w)}\right|$.
(*b*) Let G' be the Levi subgroup of P containing T and $\mathrm{B}' := \mathrm{B} \cap \mathrm{G}'$ the induced Borel. As a subset of $\pi^{-1}(\pi(e)) = \mathrm{G}'/\mathrm{B}'$, the fibre of $w^{-1}\mathrm{X}_w$ over $\pi(e) \in \mathrm{M}$ is $\phi(w)^{-1}\mathrm{X}_{\phi(w)}$.

Proof. The composite $\mathrm{B}^{\mathrm{op}} \hookrightarrow \mathrm{G} \longrightarrow \mathrm{G}/\mathrm{B}$ is an open immersion of B^{op} in $\mathrm{X} = \mathrm{G}/\mathrm{B}$. The image U ($\cong \mathrm{B}^{\mathrm{op}}$) of B^{op} in X is therefore an affine space of dimension N, whose torus-fixed coordinate rays are identified with the set Δ^- of negative roots. Restricted to U, each shifted Schubert variety $w^{-1}\mathrm{X}_w$ is the coordinate plane spanned by the coordinate vectors of the roots in $-\Phi_w$. The image of U in M is the affine space spanned by the roots in $\Delta^- \setminus \Delta_{\mathrm{P}}^-$, and the map π restricted to U is the natural projection. The image of $\pi(w^{-1}\mathrm{X}_w)$ restricted to $\pi(\mathrm{U})$ is therefore the linear space spanned by the roots in $-\Phi_w \setminus \Delta_{\mathrm{P}}$, and the fibre in U over $\pi(e)$ is the linear space spanned by $-\Phi_w \cap \Delta_{\mathrm{P}} = -\Phi_{\phi(w)}$. This establishes both (*a*) and (*b*). □

Corollary (2.4.4) — For any $w \in \mathcal{W}$ the generic fibre of $\pi|_{\mathrm{X}_w}\colon \mathrm{X}_w \longrightarrow \pi(\mathrm{X}_w)$ is $\mathrm{X}_{\phi(w)}$.

Proof. Since B acts transitively on an open subset of X_w containing $w \in \mathrm{X}$ it also acts transitively on an open subset of $\pi(\mathrm{X}_w)$ containing $\pi(w)$. Hence all fibres in this open set are isomorphic, and by Proposition 2.4.3(*b*) the fibre over $\pi(w)$ is (after shifting back) isomorphic to $\mathrm{X}_{\phi(w)}$. □

We will also use the results above in the $\{[\Omega_w]\}_{w\in\mathcal{W}}$ basis:

Proposition (2.4.5) — With notation as above, for any $w \in \mathcal{W}$,

(*a*) $\pi(\Omega_w)$ has codimension $\left|\Phi_w \setminus \Phi_{\phi(w)}\right|$ in M.
(*b*) The fibre of $(w_0w)^{-1}\Omega_w$ over $\pi(e) \in \mathrm{M}$ is $\phi(w_0w)^{-1}\Omega_{\phi(w)}$.
(*c*) The general fibre of $\pi|_{g\Omega_{w_i}} \longrightarrow \pi(g\Omega_{w_i})$ is of the class $[\Omega_{\phi(w_i)}]$.

Proof. Parts (*a*) and (*b*) are restated versions of 2.4.3(*a*) and (*b*), and (*c*) is a restated version of Corollary 2.4.4. □

Finally, we will need the result [Ri2, Theorem 1.1] of Richmond, although in slightly different notation. We include some of the details for completeness of exposition.

Corollary (2.4.6) — Suppose that $w_1, \ldots, w_k$ are elements of $\mathcal{W}$ such that $\sum \ell(w_i) = \mathrm{N}$, and let P be a parabolic subgroup, $\mathrm{M} = \mathrm{G}/\mathrm{P}$, and $\pi\colon \mathrm{X} \longrightarrow \mathrm{M}$ the projection. We further assume that $\sum_i |\Phi_{\phi_{\mathrm{P}}(w_i)}| = |\Delta_{\mathrm{P}}^+|$. Then we have the following equality of intersection numbers:

(2.4.7)
$$\bigcup_{i=1}^{k}[\Omega_{w_i}] = \left(\bigcup_{i=1}^{k}[\pi(\Omega_{w_i})]\right)\cdot\left(\bigcup_{i=1}^{k}[\Omega_{\phi(w_i)}]\right)$$

where the intersection on the left takes place in the cohomology ring $\mathrm{H}^*(\mathrm{X},\mathbf{Z})$, and the intersections on the right in $\mathrm{H}^*(\mathrm{M},\mathbf{Z})$ and $\mathrm{H}^*(\mathrm{X}',\mathbf{Z})$ respectively.

Proof. By Proposition 2.4.5(*a*) the assumption $\sum_i |\Phi_{\phi_{\mathrm{P}}(w_i)}| = |\Delta_{\mathrm{P}}^+|$ is equivalent to the condition $\sum_i \mathrm{codim}(\pi(\Omega_{w_i}),\mathrm{M}) = \dim(\mathrm{M})$. We have also assumed that $\sum \ell(w_i) = \mathrm{N}$. These are precisely the conditions [Ri2, (2), p. 3]. Since these conditions hold, one may apply [Ri2, Theorem 1.1], whose conclusion is that the equality (2.4.7) above holds. ☐

Remark. We will only apply Corollary 2.4.6 to $w_1,\ldots, w_k$ satisfying (1.1.2). In this case, the equality (2.4.7) is also the result of [Re1, Theorem 2].

2.5. Description of fibration method. Assume that $w_1,\ldots, w_k$ satisfy (1.1.2). Let $\mathrm{P}\supset\mathrm{B}$ be a parabolic, $\mathrm{M} = \mathrm{G}/\mathrm{P}$, and $\pi\colon \mathrm{X}\longrightarrow\mathrm{M}$ the projection. We also let G' be the Levi subgroup of P, $\mathrm{B}' = \mathrm{G}'\cap\mathrm{B}$ the induced Borel, and $\mathrm{X}' = \mathrm{G}'/\mathrm{B}'$ the quotient. Finally, let $\phi_{\mathrm{P}}\colon \mathcal{W}\longrightarrow\mathcal{W}_{\mathrm{P}}$ be the map of Definition 2.4.1.

Since $\Phi_{\phi(w_i)} = \Phi_{w_i}\cap\Delta_{\mathrm{P}}$, if $w_1,\ldots,w_k$ satisfy (1.1.2) then $\phi(w_i),\ldots,\phi(w_k)$ satisfy

(2.5.1)
$$\Delta_{\mathrm{P}}^+ = \bigsqcup_{i=1}^{k}\Phi_{\phi(w_i)}.$$

Thus we may apply Corollary 2.4.6 to get the equality (2.4.7) of intersection numbers. Suppose we can show that $\bigcup_{i=1}^{k}[\pi(\Omega_{w_i})] = 1$ in $\mathrm{H}^*(\mathrm{M},\mathbf{Z})$, then (2.4.7) becomes $\bigcup_{i=1}^{k}[\Omega_{w_i}] = \bigcup_{i=1}^{k}[\Omega_{\phi(w_i)}]$. Since this second intersection is taking place in $\mathrm{H}^*(\mathrm{X}',\mathbf{Z})$, and since (2.5.1) is simply condition (1.1.2) for G', we may hope that we already know that the second intersection is 1 by induction on rank.

Thus the key inductive step for the fibration method is being able to show that the appropriate intersection in $\mathrm{H}^*(\mathrm{M},\mathbf{Z})$ is 1.

2.6. Proof of Theorem 1.2.1 in type D **: Preliminaries.** We first prove, by induction, a combinatorial lemma (Lemma 2.7.1). This lemma and an elementary observation about the cohomology ring of quadrics will establish the inductive step necessary to use the fibration method.

D_n **root systems.** Let $\epsilon_1,\ \ldots,\ \epsilon_n$ be the usual basis upon which $\mathcal{W}$ operates by signed permutations with an even number of sign changes. The positive roots are elements of the form $\epsilon_p\pm\epsilon_q$ with $p<q$. The fundamental weights are $\varpi_p = \epsilon_1+\cdots+\epsilon_p$ for $1\leqslant p\leqslant n-2$, $\varpi_{n-1} = \frac{1}{2}(\epsilon_1+\epsilon_2+\cdots+\epsilon_{n-1}-\epsilon_n)$, and $\varpi_n = \frac{1}{2}(\epsilon_1+\epsilon_2+\cdots+\epsilon_{n-1}+\epsilon_n)$. We will also use D_2 ($\cong \mathrm{A}_1\times\mathrm{A}_1$) and D_3 ($\cong\mathrm{A}_3$) for the root systems defined as above with $n=2,3$.

2.7. Reductions to D_{n-1}**.** For any $p\in\{1,\ldots,n\}$, the subset of positive roots not involving ϵ_p, i.e., the set $\Delta_p^+ := \{\epsilon_r\pm\epsilon_q \mid r<q \text{ and } r,q\neq p\}$, forms the positive roots of a sub-root system of type D_{n-1}. For any element $w\in\mathcal{W}$, the intersection $\Phi_w\cap\Delta_p^+$ is both closed and coclosed in Δ_p^+, and hence is the inversion set of an element $\overline{w}$ in the D_{n-1} Weyl

group. This reduction map from $\mathcal{W}_n$ to $\mathcal{W}_{n-1}$ is not one coming from a parabolic P as in Definition 2.4.1, unless $p = 1$. Nonetheless, the reduction map exists and by construction has the property that if $w_1, \ldots, w_k$ satisfy (1.1.2) then $\overline{w}_1, \ldots, \overline{w}_k$ also satisfy (1.1.2) (i.e., $\Delta_p^+ = \sqcup_{i=1}^k \Phi_{\overline{w}_i}$)[2].

We will use this reduction map (the process of 'deleting' an ϵ_p) repeatedly in the proof of the combinatorial lemma, so it is useful to understand the reduction explicitly. We identify Δ_p^+ with the D_{n-1} root system on the basis elements $\overline{\epsilon}_1, \ldots, \overline{\epsilon}_{n-1}$ via the natural projection induced by the linear map

$$\epsilon_q \longrightarrow \begin{cases} \overline{\epsilon}_q & \text{if } q < p \\ \overline{\epsilon}_{q-1} & \text{if } q > p. \end{cases}$$

Ignoring the signs for a moment, if we let w act on $\epsilon_1, \ldots, \epsilon_{p-1}, \epsilon_{p+1}, \ldots, \epsilon_n$, then the order on the indices of resulting basis elements defines a permutation of $n-1$ objects. The idea for the reduction $w \longrightarrow \overline{w}$ is that, treating w and $\overline{w}$ as a signed permutations, the result of acting by $\overline{w}$ on $\overline{\epsilon}_1, \ldots, \overline{\epsilon}_{n-1}$ should induce the same relative order on the images as w does above, and the signs should also be the same, with the exception of the sign of $\overline{\epsilon}_{n-1}$, which may have to be switched to ensure an even number of total sign changes (i.e., if w sends ϵ_p to the negative of some basis vector).

Explicitly, if $w(\epsilon_p) = \epsilon_{p'}$ for some p' (as opposed to $w(\epsilon_p) = -\epsilon_{p'}$), then

$$\overline{w}(\overline{\epsilon}_q) = \begin{cases} \pm\overline{\epsilon}_{q'} & \text{if } q < p,\ w(\epsilon_q) = \pm\epsilon_{q'}, \text{ and } q' < p' \\ \pm\overline{\epsilon}_{q'-1} & \text{if } q < p,\ w(\epsilon_q) = \pm\epsilon_{q'}, \text{ and } q' \geqslant p' \\ \pm\overline{\epsilon}_{q'} & \text{if } q \geqslant p,\ w(\epsilon_{q+1}) = \pm\epsilon_{q'}, \text{ and } q' < p' \\ \pm\overline{\epsilon}_{q'-1} & \text{if } q \geqslant p,\ w(\epsilon_{q+1}) = \pm\epsilon_{q'}, \text{ and } q' \geqslant p' \end{cases}$$

where (for example in the first case above), the instructions mean $+\overline{\epsilon}_{q'}$ if $w(\epsilon_q) = +\epsilon_{q'}$ and $-\overline{\epsilon}_{q'}$ if $w(\epsilon_q) = -\epsilon_{q'}$.

If instead $w(\epsilon_p) = -\epsilon_{p'}$ for some p' then $\overline{w}$ is the composite of the rule above followed by the map sending $\overline{\epsilon}_{n-1}$ to $-\overline{\epsilon}_{n-1}$ and acting as the identity on each $\overline{\epsilon}_i$, for $i < n-1$ (this ensures an even number of sign changes).

For our inductive argument we will need one fact which follows from the explicit formulae for the reduction. Suppose that w is an element of $\mathcal{W}_n$ such that $w(\epsilon_1) = +\epsilon_q$ for $q < n$. If $\overline{w}$ is the result of deleting some ϵ_p with $p > 1$ then the only way that $\overline{w}(\overline{\epsilon}_1) = -\overline{\epsilon}_{q'}$ for some q', or $\overline{w}(\overline{\epsilon}_1) = \overline{\epsilon}_{n-1}$ is if $w(\epsilon_1) = +\epsilon_{n-1}$ (i.e., $q = n-1$) and $w(\epsilon_p) = \pm\epsilon_n$. In this case $\overline{w}(\overline{\epsilon}_1) = \pm\overline{\epsilon}_{n-1}$, although we will not need this detail.

We now prove the main combinatorial lemma for dealing with the D_n case.

Lemma (2.7.1) — Suppose that we are in the D_n case, and that $w_1, \ldots, w_k$ satisfy (1.1.2). Then there exists i such that $w_i(\epsilon_1) \in \{-\epsilon_1, -\epsilon_2, \ldots, -\epsilon_n, \epsilon_n\}$, i.e., for this i either $w_i(\epsilon_1) = -\epsilon_p$ for some p, or $w_i(\epsilon_1) = \epsilon_n$.

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[2] These types of maps on Weyl groups, also called *flattening maps*, have been used in other contexts. The paper [BB] and its references are good examples.

Proof. Suppose that there is a counterexample for D_n, i.e., $w_1, \ldots, w_k$ satisfying (1.1.2) such that $w_i(\epsilon_1) = \epsilon_{p_i}$, $1 \leqslant p_i \leqslant n-1$ for all $i = 1, \ldots, k$. We will show that we can always reduce such a counterexample in D_n to a counterexample in D_{n-1}. For $n \geqslant 5$ this will follow by a counting argument (and contradiction), for $n = 3, 4$ by a more detailed argument (and contradiction). Finally, it is obvious for D_2 that no such counterexample exists, and this final contradiction proves the lemma.

Case D_n, $n \geqslant 5$. We look for ϵ_p, $2 \leqslant p \leqslant n$ that we can 'delete', and still maintain the counterexample. If it is not possible to delete some ϵ_p and still maintain the counterexample, then for each p, $2 \leqslant p \leqslant n$, there must be an i_p such that $w_{i_p}(\epsilon_1) = \epsilon_{n-1}$ and $w_{i_p}(\epsilon_p) = \pm\epsilon_n$. The element w_{i_p} then inverts exactly $n-2$ positive roots involving ϵ_1 (exactly half of the positive roots $\{\epsilon_1 \pm \epsilon_q \mid q \neq p\}$). For different p, the corresponding w_{i_p} are also distinct, since (for instance) $w_{i_p}^{-1}(\epsilon_n) = \pm\epsilon_p$. Hence by (1.1.2) these elements invert $(n-1)(n-2)$ distinct positive roots involving ϵ_1. Since there are exactly $2(n-1)$ such roots, this gives the inequality $(n-1)(n-2) \leqslant 2(n-1)$ or $n \leqslant 4$. Thus if $n \geqslant 5$ there is always such an ϵ_p, and we can reduce the counterexample to the D_4 case.

Case D_4. If there is no $p \in \{2, 3, 4\}$ so that we can delete ϵ_p and preserve the counterexample, then as above there must be (reordering the w_i as necessary) w_1, w_2, w_3 such that $w_i(\epsilon_1) = \epsilon_3$ and $w_i(\epsilon_{i+1}) = \pm\epsilon_4$, $i = 1, 2, 3$. Each such w_i inverts exactly two roots involving ϵ_1, and hence we must have $w_i(\epsilon_1) = \epsilon_1$ for all $i \geqslant 4$ (if $k \geqslant 4$), since there are exactly six positive roots of the form $\epsilon_1 \pm \epsilon_q$, $q \in \{2, 3, 4\}$.

From the conditions, w_1 inverts exactly one of $\epsilon_2 \pm \epsilon_3$ and exactly one of $\epsilon_2 \pm \epsilon_4$. This implies that $w_2(\epsilon_2) = +\epsilon_q$ with $q \in \{1, 2\}$, since if $w_2(\epsilon_2) = -\epsilon_q$ then both of $\epsilon_2 \pm \epsilon_3$ would be inverted by w_2, contradicting the fact that w_1 inverts exactly one of them, and condition (1.1.2). Similarly, we must have $w_3(\epsilon_2) = +\epsilon_q$ with $q \in \{1, 2\}$ or w_3 would invert both of $\epsilon_2 \pm \epsilon_4$.

But now none of w_1, w_2, and w_3 inverts $\epsilon_1 + \epsilon_2$, and since $w_i(\epsilon_1) = \epsilon_1$ for all $i \geqslant 4$, we see that $\epsilon_1 + \epsilon_2$ is never inverted, again contradicting (1.1.2). Thus we may reduce the counterexample to the D_3 case.

Case D_3. Again, assume that there is no $p \in \{2, 3\}$ which can be deleted and maintain the counterexample. Then (after reordering) we must have $w_i(\epsilon_1) = \epsilon_2$, $w_i(\epsilon_{i+1}) = \pm\epsilon_3$ for $i = 1, 2$, and $w_i(\epsilon_1) = \epsilon_1$ for $i \geqslant 3$. Again w_1 inverts exactly one of $\epsilon_2 \pm \epsilon_3$, so condition (1.1.2) implies that we must have $w_2(\epsilon_2) = \epsilon_1$. But now, as before, no w_i inverts $\epsilon_1 + \epsilon_2$, a contradiction. Thus we can reduce any counterexample in D_3 to D_2.

Case D_2. The condition for the counterexample now means that $w_i(\epsilon_1) = \epsilon_1$ for all i, and hence $w_i(\epsilon_2) = \epsilon_2$ for all i (since each w_i is a signed permutation of ϵ_1, ϵ_2 with an even number of sign changes). I.e., each $w_i = e$. This certainly contradicts (1.1.2), and hence no such counterexample exists.

This finishes the proof of Lemma 2.7.1. $\square$

2.8. Intersections on even dimensional quadrics. Let P_1 be the parabolic so that $\mathcal{W}_{\mathrm{P}_1} \subset \mathcal{W}$ is the stabilizer of ϵ_1. Then $\mathrm{Q}_n := \mathrm{G}/\mathrm{P}_1$ is a smooth quadric hypersurface in $\mathbf{P}^{2n-1}$. Let $\pi\colon \mathrm{X} \longrightarrow \mathrm{Q}_n$ be the projection. The fibre $\mathrm{X}' = \pi^{-1}(\pi(e))$ is of type D_{n-1}.

The cohomology ring of Q_n is generated by h (the class of a hyperplane section) and two classes a and b of codimension $(n-1)$ (i.e., in the middle cohomology), satisfying the relations

(2.8.1) $\quad h^{n-1} = a + b, ha = hb, h^n a = 0, a^2 = b^2 = \frac{1}{2}(1-(-1)^n)[pt], ab = \frac{1}{2}(1+(-1)^n)[pt]$

where $[pt]$ is the class of a point (see, for instance [L, Theorem 2]). The classes a and b are represented by linear subspaces of $\mathbf{P}^{2n-1}$ of dimension $n-1$ contained in Q_n.

The cohomology ring therefore has the presentation

$$H^*(Q_n, \mathbf{Z}) = \frac{\mathbf{Z}[h,a,b]}{\text{(relations in (2.8.1))}}.$$

We will use the integral basis for $H^*(Q_n, \mathbf{Z})$ given by $\{h^k\}_{0 \leqslant k \leqslant n-2}$ in codimension $\leqslant n-2$, a and b in codimension $n-1$, and $\{h^k a\}_{1 \leqslant k \leqslant n-1}$ in codimensions n to $2(n-1)$. Under the projection π, the image of each Schubert cell in X is sent to a variety whose cohomology class is one of the integral basis classes above. The complex codimension of the image of Ω_w is the number of roots involving ϵ_1 (the roots of the form $\epsilon_1 \pm \epsilon_q$) in Φ_w.

Since Q_n has degree 2, $h^{2n-2} = 2[pt]$, and since a is the class of a linear space $h^{n-1}a = 1[pt]$. Given our choice of basis classes, this immediately proves the following result.

Lemma (2.8.2) — Let $c_1, \ldots, c_k$ be basis cohomology classes in $H^*(Q_n, \mathbf{Z})$ whose (complex) codimensions sum to $2(n-1) = \dim(Q_n)$. Then

$$\bigcup_{i=1}^{k} c_i = \begin{cases} 1 & \text{if some } c_i \text{ has codimension} \geqslant n-1 \\ 2 & \text{if all } c_i \text{ have codimension} \leqslant n-2. \end{cases}$$

2.9. Proof of Theorem 1.2.1 in type D : Geometric Approach. We prove the result by induction on n. The case $n=3$ is $D_3 = A_3$, which is covered by §2.3. It therefore suffices to give the inductive step. Suppose that $w_1, \ldots, w_k$ satisfy (1.1.2), then by Lemma 2.7.1 there is some i so that $w_i(\epsilon_1) \in \{-\epsilon_1, \ldots, -\epsilon_n, \epsilon_n\}$. For such an i, w_i inverts at least $n-1$ positive roots involving ϵ_1 (the roots of the form $\epsilon_1 \pm \epsilon_q$). Hence $\pi(\Omega_{w_i})$ has codimension $\geqslant n-1$ in Q_n by Proposition 2.4.5(*a*). By Lemma 2.8.2 this means that $\bigcup_{i=1}^k [\pi(\Omega_{w_i})] = 1$, and hence by Corollary 2.4.6 that

$$\bigcup_{i=1}^{k} [\Omega_{w_i}] = \bigcup_{i=1}^{k} [\Omega_{\phi(w_i)}],$$

where ϕ is the map $\phi \colon \mathcal{W} \longrightarrow \mathcal{W}_{P_1}$ of Definition 2.4.1. Since $\phi(w_1), \ldots, \phi(w_k)$ are elements of the D_{n-1} root system satisfying (1.1.2), we conclude by the inductive hypothesis that we have $\bigcup_{i=1}^k [\Omega_{w_i}] = 1$. □

It is also possible to use the method of §2.2 to prove Theorem 1.2.1 in the D_n case; a key step is again Lemma 2.7.1. To avoid some extra combinatorial digressions, we only sketch the argument.

2.10. Proof of Theorem 1.2.1 in type D **: Combinatorial Approach.** Let u be such that $w_i \leqslant w_i u$ for $i = 1,\ldots, k$. We want to show that $u = e$. We first show that $u(\epsilon_1) = \epsilon_1$. If $u(\epsilon_1) = \pm\epsilon_q$ with $q > 1$ then $\mu = \varpi_1 - u\varpi_1 = \epsilon_1 \mp \epsilon_q$ is a positive root, contradicting Lemma 2.2.6(*d*). If $u(\epsilon_1) = -\epsilon_1$, then $\mu := \varpi_1 - u\varpi_1 = 2\epsilon_1$, which is a sum of positive roots. However, by Lemma 2.7.1 there is an i such that $w_i(\epsilon_1) = -\epsilon_p$ or $w_i(\epsilon_1) = \epsilon_n$. Then $w_i\mu = -2\epsilon_p$ or $w_i\mu = 2\epsilon_n$, neither of which are sums of positive roots, contradicting Lemma 2.2.6(*c*). Thus we must have $u(\epsilon_1) = \epsilon_1$, and so $u \in \mathcal{W}_{\mathrm{P}_1}$. Applying the map $\phi\colon \mathcal{W} \longrightarrow \mathcal{W}_{\mathrm{P}_1}$ one can check (these details are omitted) that $\phi(w_i) \leqslant \phi(w_i)\phi(u)$, where the order is now the Bruhat order on $\mathcal{W}_{\mathrm{P}_1}$. By induction, the only solution is $\phi(u) = e$, and since $u \in \mathcal{W}_{\mathrm{P}}$, this implies that $u = e$. □

2.11. Proof of Theorem 1.2.1 in type G_2**.** The argument is elementary for any rank 2 group; by Lemma 2.11.1 below there are at most two w_i with $w_i \neq e$. If there are exactly two such w_i, say w_1 and w_2 then condition (1.1.2) implies that Ω_{w_1} and Ω_{w_2} are Poincaré dual pairs, so $\bigcup_{i=1}^{k}[\Omega_{w_i}] = [\Omega_{w_1}] \cup [\Omega_{w_2}] = 1$. If there is only one such w_i, then it must be equal to w_0 and since Ω_{w_0} is a point we again have $\bigcup_{i=1}^{k}[\Omega_{w_i}] = 1$.

Lemma (2.11.1) — If $w_1, \ldots, w_k \in \mathcal{W}$ satisfy (1.1.2) then the number of w_i which are not equal to e is at most $\mathrm{rank}(\mathrm{G})$.

Proof. Each Φ_{w_i} is coclosed, so if Φ_{w_i} does not contain any simple roots, then Φ_{w_i} does not contain any roots and therefore $\Phi_{w_i} = \varnothing$ and so $w_i = e$. Therefore if $w_i \neq e$ the set Φ_{w_i} contains a simple root. Since the union $\Delta^+ = \sqcup_i \Phi_{w_i}$ is disjoint, the number of w_i with $w_i \neq e$ is therefore at most the number of simple roots. □

3. APPENDIX : THE EQUIVALENCE OF CONDITIONS (*iii*) AND (*iv*)

It is clear that (*iii*) (being the case $k = 3$ of (*iv*)) is implied by (*iv*). To prove the other direction we will need to discuss the product $\odot_0$ in more detail. This product is obtained by specializing a deformation of the ordinary cup product. This deformation was introduced by Belkale and Kumar.

3.1. The Belkale-Kumar deformation of the cup product on G/B**.** Let $\alpha_1,\ldots, \alpha_n$ denote the simple roots of G and let $\mathcal{Q}$ be the root lattice. Introduce variables $\tau_1,\tau_2,\ldots, \tau_n$, one for each simple root. For any $\gamma \in \mathcal{Q}$ we use the notation $\underline{\tau}^{\gamma}$ to denote the Laurent monomial $\tau_1^{m_1}\tau_2^{m_2}\cdots\tau_n^{m_n}$, where $\sum_{i=1}^{n} m_i\alpha_i = \gamma$ is the unique expression of γ as a $\mathbf{Z}$-linear combination of simple roots.

Following [BK, Definition 5] for any $w \in \mathcal{W}$ we define $\chi_w = \sum_{\alpha\in\Phi_w}\alpha$. The operation $\odot$ acting on two basis classes is defined [BK, p. 199] by the formula

$$[\Omega_{w_1}] \odot [\Omega_{w_2}] := \sum_{w} \underline{\tau}^{(\chi_w-\chi_{w_1}-\chi_{w_2})} c^{w}_{w_1,w_2}[\Omega_w].$$

Belkale and Kumar [BK, Proposition 17(a)] prove that if $c^{w}_{w_1,w_2} \neq 0$ then $\chi_w - \chi_{w_1} - \chi_{w_2}$ is in the positive root lattice, and thus all exponents of $\underline{\tau}^{(\chi_w - \chi_{w_1} - \chi_{w_2})}$ are nonnegative. The product above therefore takes values in $\mathrm{H}^*(\mathrm{X}, \mathbf{Z}) \otimes \mathbf{Z}[\tau_1, \ldots, \tau_n]$. The operation $\odot$ is then extended to all of $\mathrm{H}^*(\mathrm{X}, \mathbf{Z}) \otimes \mathbf{Z}[\tau_1, \ldots, \tau_n]$ by $\mathbf{Z}[\tau_1, \ldots, \tau_n]$-linearity.

From the formula it is clear that $\odot$ is commutative. One checks by induction (see [BK, Proposition 17(c)]) that for any $w_1, \ldots, w_k \in \mathcal{W}$

(3.1.1) $$[\Omega_{w_1}] \odot [\Omega_{w_2}] \odot \cdots \odot [\Omega_{w_k}] = \sum_{w \in \mathcal{W}} \underline{\tau}^{(\chi_w - \sum \chi_{w_i})} c^{w}_{w_1,\ldots,w_k} [\Omega_w],$$

where $c^{w}_{w_1,\ldots,w_k}$ is the coefficient of $[\Omega_w]$ in the expression of $\cup_{i=1}^{k} [\Omega_{w_i}]$ as a sum of basis classes. Thus the product of basis elements in the deformed product is the usual cup product, with each term in the result shifted by a monomial in $\tau_1, \ldots, \tau_n$, where the monomial depends on the term and the classes being multiplied.

Setting all $\tau_i = 1$ recovers the usual cup product. The Belkale-Kumar product $\odot_0$ is defined as the specialization obtained by setting all $\tau_i = 0$.

Lemma (3.1.2) — If $w_1, \ldots, w_k \in \mathcal{W}$ satisfy (1.1.2), then

(*a*) $[\Omega_{w_1}] \odot_0 [\Omega_{w_2}] \odot_0 \cdots \odot_0 [\Omega_{w_k}] = [\Omega_{w_1}] \cup [\Omega_{w_2}] \cup \cdots \cup [\Omega_{w_k}]$.

(*b*) For any subset $\mathrm{I} \subseteq \{1, 2, \ldots, k\}$ there is an element $w \in \mathcal{W}$ so that $\Phi_w = \sqcup_{i \in \mathrm{I}} \Phi_{w_i}$.

Proof. One of the properties of the inversion sets is that for any $w \in \mathcal{W}$, $\ell(w) = |\Phi_w|$. Therefore if $w_1, \ldots, w_k$ satisfy (1.1.2) we have $\sum_{i=1}^{k} \ell(w_i) = \sum_{i=1}^{k} |\Phi_{w_i}| = |\Delta^+| = \dim(\mathrm{X})$. The only class in dimension zero is the class of a point, $[\Omega_{w_0}]$. Since (again by (1.1.2)) we have $\chi_{w_0} = \sum_{i=1}^{k} \chi_{w_i}$, we conclude by (3.1.1) that

$$[\Omega_{w_1}] \odot [\Omega_{w_2}] \odot \cdots \odot [\Omega_{w_k}] = c^{w_0}_{w_1,\ldots,w_k} [\Omega_{w_0}] = \bigcup_{i=1}^{k} [\Omega_{w_k}].$$

This proves (*a*).

For a proof of part (*b*), see [DR, Corollary 5.4.9] or [D-W, Proposition 2.7] (the proof in the second reference is presented in a more combinatorial context, and in the language of type A, but works in all types). Part (*b*) may also be deduced using Lie algebra cohomology. □

3.2. Proof that (iii) implies (iv). By Lemma 3.1.2(*a*) it is sufficient to show that $[\Omega_{w_1}] \odot_0 \cdots \odot_0 [\Omega_{w_k}] = 1$. By part (*b*) of the same lemma there is an element $u \in \mathcal{W}$ such that $\Phi_u = \Phi_{w_{k-1}} \sqcup \Phi_{w_k}$. By (1.1.1) and (*iii*) (in its equivalent form (*ii*)) we have $[\Omega_{w_{k-1}}] \odot_0 [\Omega_{w_k}] = 1[\Omega_u]$. Thus

$$[\Omega_{w_1}] \odot_0 [\Omega_{w_2}] \odot_0 \cdots \odot_0 [\Omega_{w_{k-2}}] \odot_0 [\Omega_{w_{k-1}}] \odot_0 [\Omega_{w_k}] = [\Omega_{w_1}] \odot_0 [\Omega_{w_2}] \odot_0 \cdots \odot_0 [\Omega_{w_{k-2}}] \odot_0 [\Omega_u]$$

with $(\bigsqcup_{i=1}^{k-2} \Phi_{w_i}) \sqcup \Phi_u = \Delta^+$. I.e., we have reduced the expression we are interested in to a similar expression with one fewer term. Continuing in this manner we reduce the expression to $[\Omega_{w_0}]$, the class of a point. □

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Department of Mathematics and Statistics, Queen's University, Kingston, Ontario, K7L 3N6, Canada

E-mail address: dimitrov@queensu.ca

E-mail address: mike.roth@queensu.ca