DECOMPOSING INVERSION SETS OF PERMUTATIONS AND APPLICATIONS TO FACES OF THE LITTLEWOOD-RICHARDSON CONE

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Abstract. If $\alpha \in S_n$ is a permutation of $\{1, 2, \dots, n\}$, the inversion set of α is $\Phi(\alpha) = \{(i, j) \mid 1 \leqslant i < j \leqslant n, \alpha(i) > \alpha(j)\}$. We describe all r-tuples $\alpha_1, \alpha_2, \dots, \alpha_r \in S_n$ such that $\Delta_n^+ = \{(i, j) \mid 1 \leqslant i < j \leqslant n\}$ is the disjoint union of $\Phi(\alpha_1), \Phi(\alpha_2), \dots, \Phi(\alpha_r)$. Using this description we prove that certain faces of the Littlewood-Richardson cone are simplicial and provide an algorithm for writing down their sets of generating rays. We also discuss analogous problems for the Weyl groups of root systems of types B, C and D providing solutions for types B and C. Finally we provide some enumerative results and introduce a useful tool for visualizing inversion sets.
Keywords: Inversion set, Simple permutation, Littlewood-Richardson cone, Catalan numbers.

1. Introduction

1.1. Given a positive integer n, we set

$$\Delta_n^+ := \{(i, j) \mid 1 \leqslant i < j \leqslant n\}.$$

In accordance with terminology from Lie Theory, we will refer to the elements of Δ_n^+ as *positive roots*, the element $(1, n)$ is the *highest root*, and the elements $(i, i + 1)$ with $1 \leqslant i \leqslant n - 1$ are the *simple roots*.

We describe an element $\alpha \in S_n$ as a function, writing $\alpha = (\alpha(1), \alpha(2), \dots, \alpha(n))$, and define *the inversion set of* α, $\Phi(\alpha)$, by

$$\Phi(\alpha) := \{(i, j) \in \Delta_n^+ \mid \alpha(i) > \alpha(j)\}.$$

We use I_n for the identity permutation: $I_n = (1, 2, \dots, n)$, and J_n for the longest element: $J_n = (n, n - 1, \dots, 1) \in S_n$. Note that $\Phi(I_n) = \emptyset$ and $\Phi(J_n) = \Delta_n^+$. It is not hard to see that the element $\alpha \in S_n$ is determined by its inversion set $\Phi(\alpha)$. Thus there are exactly $n!$ subsets of Δ_n^+ which are inversion sets.

Throughout the paper we use the following notational conventions. We use the symbol $\sqcup$ to denote a disjoint union. Often we will write I, J, Δ^+, etc. instead of I_n, J_n, Δ_n^+, etc. when the value of n is clear from the context.

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Definition 1.1. A *decomposition* of an inversion set $\Phi(\alpha)$ is a set of disjoint inversion sets $\Phi(\alpha_1), \Phi(\alpha_2), \ldots, \Phi(\alpha_r)$ such that

$$\Phi(\alpha) = \Phi(\alpha_1) \sqcup \Phi(\alpha_2) \sqcup \cdots \sqcup \Phi(\alpha_r) \ .$$

Note that $\Phi(I) = \emptyset$ may occur in a decomposition. The decomposition is called *trivial* if $\Phi(\alpha) = \Phi(\alpha_a)$ for some a with $1 \leqslant a \leqslant r$, and hence that $\alpha_i = I$ for all $i \neq a$.

We say that an element $\alpha \in S_n$ (and its inversion set $\Phi(\alpha)$) is *reducible* if there exists a non-trivial decomposition of $\Phi(\alpha)$. Otherwise we say that α (and $\Phi(\alpha)$) is *irreducible*[1]. We call a decomposition as above an *irreducible decomposition* if each $\Phi(\alpha_i)$ is irreducible.

Solving the following problem was the motivation for this article.

Problem 1.2. Describe all decompositions of Δ_n^+:

$$\Delta_n^+ = \Phi(\alpha_1) \sqcup \Phi(\alpha_2) \sqcup \cdots \sqcup \Phi(\alpha_r),$$

The order of the inversion sets in the decomposition is irrelevant.

1.2. We are interested in this problem because of its relation to two other problems:

(i) determining the regular codimension n faces of the Littlewood-Richardson cone;
(ii) studying the cup product of the cohomology of line bundles on homogeneous varieties.

We briefly describe these two problems in the next paragraphs.

The Littlewood-Richardson cone. If A is a Hermitian matrix, denote by $\lambda = (\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_n) \in \mathbb{R}^n$ its eigenvalues and let $\mathbb{R}^{3n}_+ = \{(\lambda, \mu, \nu) \,|\, \lambda_i \geqslant \lambda_{i+1}, \mu_i \geqslant \mu_{i+1}, \nu_i \geqslant \nu_{i+1} \text{ for } 1 \leqslant i \leqslant n-1\}$. In 1912 H. Weyl posed the following question: For which triples $(\lambda, \mu, \nu) \in \mathbb{R}^{3n}_+$ do there exist Hermitian matrices A, B, C such that $C = A + B$ and whose eigenvalues are λ, μ, ν respectively. In 1962 A. Horn proved that the set of such triples is a polyhedral cone $\mathcal{C}'$ and conjectured inequalities determining $\mathcal{C}'$. Horn's conjecture was proved in the 1990's by Klyachko and Knutson and Tao, see [F] for a nice exposition on Horn's conjecture. It is worth mentioning that the lattice points of $\mathcal{C}'$ are exactly the triples (λ, μ, ν) for which the corresponding Littlewood-Richardson coefficient $c^{\nu}_{\lambda,\mu}$ is nonzero. It is often convenient to study the cone $\mathcal{C}''$ corresponding to the relation $A + B + C = 0$ instead of $\mathcal{C}'$ corresponding to $C = A + B$ thus symmetrizing the roles of λ, μ, and ν. N. Ressayre [R] described all *regular faces of* $\mathcal{C}''$, i.e. faces that intersect the interior of $\mathbb{R}^{3n}_+$. The regular faces of codimension n are in a bijection with triples $\alpha_1, \alpha_2, \alpha_3$ of elements of S_n with the property that $\Delta_n^+ = \Phi(\alpha_1) \sqcup \Phi(\alpha_2) \sqcup \Phi(\alpha_3)$: each regular face of codimension n is the intersection of $\mathbb{R}^{3n}_+$ with the subspace of codimension n defined by

$$\alpha_1^{-1}\lambda + \alpha_2^{-1}\mu + \alpha_3^{-1}\nu = 0$$

[1] In keeping with standard usage in algebra, a better term for these kinds of permutations would be *indecomposable*. However, the term "indecomposable permutation" is already established in the literature for a different class of permutations. Moreover, following [AAK], we use the terms "plus- or minus-indecomposibles" in relation to the inflation procedure in §1.3, and wish to avoid a conflict of terminology.

for the corresponding triple $\alpha_1, \alpha_2, \alpha_3$.

Cup products of line bundles on homogeneous varieties. Let $G = GL_n(\mathbb{C})$, let $B \subset G$ be a Borel subgroup, and let $X = G/B$. The Picard group of G-equivariant line bundles on X is isomorphic to $\mathbb{Z}^n$. We denote by $\mathcal{L}_\lambda$ the line bundle on X which corresponds to the B-character $-\lambda$. We call $\lambda \in \mathbb{Z}^n$ *dominant* if $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_n$ and *strictly dominant* if $\lambda_1 > \lambda_2 > \ldots > \lambda_n$. Let $\rho = (n-1, n-2, \ldots, 0) \in \mathbb{Z}^n$. We call $\lambda \in \mathbb{Z}^n$ *regular* if there exists $\alpha \in S_n$ such that $\alpha \cdot \lambda := \alpha(\lambda + \rho) - \rho$ is dominant. Such an element α is uniquely determined by λ and we denote it by α_λ. The celebrated Borel-Weil-Bott theorem calculates the cohomology groups $H^q(X, \mathcal{L}_\lambda)$. In particular, it states that $H^q(X, \mathcal{L}_\lambda)$ is zero unless λ is regular and q equals the length of α_λ. In this case, $H^q(X, \mathcal{L}_\lambda) \cong V(\alpha_\lambda \cdot \lambda)^*$, where for any dominant weight μ, $V(\mu)$ denotes the irreducible G-module with highest weight μ. In [DR] the following question was studied: For which pairs $\lambda, \mu \in \mathbb{Z}^n$ is the cup product map

$$H^{q_1}(X, \mathcal{L}_\lambda) \otimes H^{q_2}(X, \mathcal{L}_\mu) \xrightarrow{\cup} H^{q_1+q_2}(X, \mathcal{L}_{\lambda+\mu})$$

nonzero provided that the all cohomology groups above are nonzero? Theorem I in [DR] states that the cup-product map above is non-zero if and only if $\Phi(\alpha_{\lambda+\mu}) = \Phi(\alpha_\lambda) \sqcup \Phi(\alpha_\mu)$. This is equivalent (see Lemma 2.5 below) to the condition that $\Delta_n^+ = \Phi(\alpha_\lambda) \sqcup \Phi(\alpha_\mu) \sqcup \Phi(J_n \alpha_{\lambda+\mu})$.

Both of these motivating problems have versions involving an arbitrary number of factors, (i.e., the sum of r matrices, or the cup product of r cohomology groups), and their solutions are similarly expressed as decompositions of Δ_n with $r+1$ factors. We were thus led to consider Problem 1.2.

1.3. Before we state the main results of the paper, we introduce some concepts and state background results.

Definition 1.3. An *interval* (of size t) is a set of consecutive integers $\{i, i+1, i+2, \ldots, i+t-1\}$. For a permutation $\alpha \in S_n$, a *block* (of size t) of α is an interval $\{i, i+1, i+2, \ldots, i+t-1\}$ of size t such that the set $\{\alpha(i), \alpha(i+1), \ldots, \alpha(i+t-1)\}$ is also an interval (of size t). Every permutation in S_n has n blocks of size 1 and a block of size n. If $\alpha \in S_n$ has no blocks of size t for all $1 < t < n$ then we say that α is *simple*[2].

Example 1.4. The permutation $(9, 7, 1, 5, 3, 4, 6, 8, 2) \in S_9$ has a block of size 8 corresponding to the interval $\{2, 3, 4, 5, 6, 7, 8, 9\}$ and a block size 4 corresponding to the interval $\{4, 5, 6, 7\}$. The permutation $(5, 2, 6, 1, 4, 7, 3) \in S_7$ has no non-trivial blocks and so is simple.

To state our results we need to introduce an inflation procedure to describe permutations inductively. We describe this procedure heuristically as follows. We consider a permutation on n letters as a shuffling of a deck of n cards. To shuffle, we first cut the deck into m piles

[2] We warn the reader that some authors use the terminology *connected* rather than simple.

of sizes $z_1, z_2, \ldots, z_m$ respectively. Shuffle each of these piles according to a permutation $\beta_i \in S_{z_i}$. Finally reassemble the piles in an order determined by a permutation $\sigma \in S_m$. The resulting permutation in S_n is denoted by $\sigma[\beta_1, \beta_2, \ldots, \beta_m]$ and is called an *inflation of* σ *by* $\beta_1, \beta_2, \ldots, \beta_m$.

Example 1.5. We evaluate the inflation $\alpha = (3,1,4,2)[(1,2,3),(1),(1,2),(2,1)]$. Here $m = 4$, and from the sizes of $\beta_1, \ldots, \beta_4$ we see that α is an element of S_n, with $n = 3+1+2+2 = 8$. To compute α, we start by splitting $1, \ldots, 8$ into blocks of sizes 3, 1, 2, and 2 : $\boxed{123}-\boxed{4}-\boxed{56}-\boxed{78}$. We then apply each β_i to the i-th block to get $\boxed{123}-\boxed{4}-\boxed{56}-\boxed{87}$. Finally, we permute each of the blocks as specified by $\sigma = (3,1,4,2)$ to get $\boxed{4}-\boxed{87}-\boxed{123}-\boxed{56}$. The resulting permutation sends 1 to the 4th position, 2 to the 5th position, 3 to the 6th position, and so on, finally sending 8 to the 2nd position. In function notation, this is the permutation $\alpha = (4,5,6,1,7,8,3,2)$.

For a formal characterization of inflation see §3; for a history of the inflation procedure and a discussion of a number of applications we refer the reader to the survey article of Brignall [Br]. The definition of inflation in Brignall's article [Br, §1.1] is equivalent to the characterization in §3, and to the "shuffling cards" description above. A graphical description of inflation in terms of inversion sets appears in §A.3.

Note that a permutation $\alpha \in S_n$ is simple if and only if α cannot be expressed as an inflation $\alpha = \sigma[\beta_1, \beta_2, \ldots, \beta_m]$ with $\sigma \in S_m$ and $2 \leqslant m \leqslant n-1$.

Definition 1.6. A permutation $\alpha \in S_n$ is called *plus-decomposable* if α may be written in the form $\alpha = I_2[\beta_1, \beta_2]$. Otherwise α is *plus-indecomposable*. Similarly, $\alpha \in S_n$ is called *minus-decomposable* if α may be written in the form $\alpha = J_2[\beta_1, \beta_2]$. Otherwise α is *minus-indecomposable*.

We follow [AAK] in using the terms "plus-indecomposable" and "minus-indecomposable". These are called "sum indecomposable" and "skew indecomposable", respectively, in [Br]. It is not difficult to verify that $\alpha \in S_n$ cannot be both plus-decomposable and minus-decomposable. On the other hand, there are permutations which are both plus-indecomposable and minus-indecomposable, e.g. every simple $\alpha \in S_n$ with $n > 2$.

The following theorem of Albert, Atkinson and Klazar illustrates the importance of simple permutations and the inflation procedure.

Theorem 1.7 ([AAK, Theorem 1])**.** *Let $n \geqslant 2$. For every permutation $\alpha \in S_n$ there exists a simple permutation $\sigma \in S_m$ and permutations $\beta_1, \beta_2, \ldots, \beta_m$ such that $\alpha = \sigma[\beta_1, \beta_2, \ldots, \beta_m]$. Moreover if $\sigma \neq I_2$ and $\sigma \neq J_2$ then $\beta_1, \beta_2, \ldots, \beta_m$ and σ are unique. If $\sigma = I_2$ then β_1, β_2 and σ are unique if we add the additional condition that β_1 is plus-indecomposable. Similarly, if $\sigma = J_2$ then β_1, β_2 and σ are unique if we add the additional condition that β_1 is minus-indecomposable.* $\square$

For our purposes, we modify the statement of the above theorem as follows.

Theorem 1.8. *Let $n \geqslant 2$. For every permutation $\alpha \in S_n$ there exists a permutation $\sigma \in S_m$ and permutations $\beta_1, \beta_2, \dots, \beta_m$ such that $\alpha = \sigma[\beta_1, \beta_2, \dots, \beta_m]$ where either σ is simple and $m \geqslant 4$ or $\sigma = I_m$ or $\sigma = J_m$. Furthermore this expression for α is unique if we require that m be maximal when $\sigma = I_m$ or $\sigma = J_m$, i.e., that each β_b is plus-indecomposable when $\sigma = I$ and each β_b is minus-indecomposable when $\sigma = J$.* □

Corollary 1.9. *In the notation of Theorem 1.8 above $\sigma = I_m$, $\sigma = J_m$ or σ is simple and $m \geqslant 4$ if and only if α is plus-decomposable, α is minus-decomposable or α is both plus-indecomposable and minus-indecomposable respectively.*

Definition 1.10. We say that α is expressed in *simple form* when we write $\alpha = \sigma[\beta_1, \beta_2, \dots, \beta_m]$ in the form guaranteed by Theorem 1.8, i.e, when σ is simple with $m \geqslant 4$ or $\sigma = J_m$ or $\sigma = I_m$ with m maximal.

1.4. We are now ready to state our main result. The inflation procedure described above allows us to inductively construct decompositions of Δ_n^+ into inversion sets. Assume that the set $\{1, 2, \dots, n\}$ is partitioned into m intervals of lengths $z_1, z_2, \dots, z_m$ and let $\sigma_a \in S_m$, and $\beta_{ab} \in S_{z_b}$ $(1 \leqslant a \leqslant r,\ 1 \leqslant b \leqslant m)$ be such that

$$\begin{array}{rcl} \Delta_m^+ &=& \Phi(\sigma_1) \sqcup \Phi(\sigma_2) \sqcup \ldots \sqcup \Phi(\sigma_r), \\ \\ \Delta_{z_1}^+ &=& \Phi(\beta_{11}) \sqcup \Phi(\beta_{21}) \sqcup \cdots \sqcup \Phi(\beta_{r1}), \\ \Delta_{z_2}^+ &=& \Phi(\beta_{12}) \sqcup \Phi(\beta_{22}) \sqcup \cdots \sqcup \Phi(\beta_{r2}), \\ &\vdots& \\ \Delta_{z_m}^+ &=& \Phi(\beta_{1m}) \sqcup \Phi(\beta_{2m}) \sqcup \cdots \sqcup \Phi(\beta_{rm}). \end{array}$$

Define $\alpha_1, \alpha_2, \dots \alpha_r \in S_n$ by

$$\begin{array}{rcl} \alpha_1 &=& \sigma_1[\beta_{11}, \beta_{12}, \dots, \beta_{1m}], \\ \alpha_2 &=& \sigma_2[\beta_{21}, \beta_{22}, \dots, \beta_{2m}], \\ &\vdots& \\ \alpha_r &=& \sigma_r[\beta_{r1}, \beta_{r2}, \dots, \beta_{rm}]. \end{array}$$

It then follows that

$$\Delta_n^+ = \Phi(\alpha_1) \sqcup \Phi(\alpha_2) \sqcup \cdots \sqcup \Phi(\alpha_r).$$

Our main result below is that every decomposition of Δ_n^+ into inversion sets can be constructed in such a way. Moreover, we identify a canonical way of representing $\alpha_1, \alpha_2, \dots, \alpha_r$ as inflations which identifies the decomposition $\Delta_n^+ = \Phi(\alpha_1) \sqcup \Phi(\alpha_2) \sqcup \cdots \sqcup \Phi(\alpha_r)$ uniquely.

Theorem 1.11. *Suppose $\alpha_1, \alpha_2, \dots, \alpha_r \in S_n$ and*

$$\Delta_n^+ = \Phi(\alpha_1) \sqcup \Phi(\alpha_2) \sqcup \cdots \sqcup \Phi(\alpha_r)$$

with all $\Phi(\alpha_a) \neq \emptyset$. Without loss of generality assume that the highest root $(1, n) \in \Phi(\alpha_1)$. Let $\alpha_1 = \sigma_1[\beta_{11}, \beta_{12}, \dots, \beta_{1m}]$ be the simple form expression of α_1 with a corresponding *partition of the set* $\{1,2,\ldots,n\}$ *into* m *intervals of lengths* $z_1,z_2,\ldots,z_m$. *Then, up to reordering of* $\alpha_2,\alpha_3,\ldots,\alpha_r$, *there exists a unique set of elements* $\sigma_a \in S_m$ *and* $\beta_{ab} \in S_{z_b}$ *such that* $\alpha_a = \sigma_a[\beta_{a1},\beta_{a2},\ldots,\beta_{am}]$, *for* $a=2,\ldots,r$, $b=1,2,\ldots,m$ *and*

(i)

$$\begin{aligned}\Delta_m^+ &= \Phi(\sigma_1)\sqcup\Phi(\sigma_2)\sqcup\ldots\sqcup\Phi(\sigma_r),\\ \\ \Delta_{z_1}^+ &= \Phi(\beta_{11})\sqcup\Phi(\beta_{21})\sqcup\cdots\sqcup\Phi(\beta_{r1}),\\ \Delta_{z_2}^+ &= \Phi(\beta_{12})\sqcup\Phi(\beta_{22})\sqcup\cdots\sqcup\Phi(\beta_{r2}),\\ &\ \vdots\\ \Delta_{z_m}^+ &= \Phi(\beta_{1m})\sqcup\Phi(\beta_{2m})\sqcup\cdots\sqcup\Phi(\beta_{rm});\end{aligned}$$

(ii) *if* α_1 *is minus-decomposable then* $\sigma_1 = J_m$ *and* $\sigma_2=\sigma_3=\cdots=\sigma_r=I_m$*;*
(iii) *if* α_1 *is minus-indecomposable then* σ_1 *is simple and* $\sigma_2 = J\sigma_1$, *and* $\sigma_3=\sigma_4=\cdots=\sigma_r=I_m$.

In particular, σ_1 *and at most one other of the* σ_a *are not the identity.*
Let q *denote the number of* σ_a *which are not* I_m*, i.e.,* $q := \begin{cases}1, & \text{if } \alpha_1 \text{ is minus-decomposable};\\ 2, & \text{if } \alpha_1 \text{ is minus-indecomposable.}\end{cases}$

Then, the above decomposition of Δ_n^+ *is irreducible if and only if the following four conditions hold*

(i) *each of the decompositions* $\Delta_{z_b}^+ = \Phi(\beta_{1b})\sqcup\Phi(\beta_{2b})\sqcup\cdots\sqcup\Phi(\beta_{rb})$ *is irreducible;*
(ii) *exactly one of of* $\beta_{a1},\beta_{a2},\ldots,\beta_{am}$ *is not equal to the identity for* $a=q+1,\ldots,r$*;*
(iii) $\beta_{ab} = I_{z_b}$ *for* $a=1,\ldots,q$ *and* $b=1,\ldots,m$*;*
(iv) $m=2$ *if* α_1 *is minus-decomposable.*

Example 1.12. Let $n=8$ and let $\alpha_1=(4,5,6,1,7,8,3,2)$, $\alpha_2=(5,3,4,8,1,2,6,7)$, $\alpha_3=(1,3,2,4,6,5,7,8)$. Here $q=2$ since α_1 is minus-indecomposable. Then $\Delta_8^+=\Phi(\alpha_1)\sqcup\Phi(\alpha_2)\sqcup\Phi(\alpha_3)$, $m=4$, and

$$\begin{aligned}\alpha_1 &= (3,1,4,2)[(1,2,3),(1),(1,2),(2,1)]\\ \alpha_2 &= (2,4,1,3)[(3,1,2),(1),(1,2),(1,2)]\\ \alpha_3 &= (1,2,3,4)[(1,3,2),(1),(2,1),(1,2)].\end{aligned}$$

Note that $\beta_{11}=I_3$, $\beta_{13}=\beta_{23}=I_2$ and $\beta_{24}=\beta_{34}=I_2$. Consequently, $\Delta_3^+=\Phi(\beta_{11})\sqcup\Phi(\beta_{21})\sqcup\Phi(\beta_{31})=\Phi(\beta_{21})\sqcup\Phi(\beta_{31})$, $\Delta_2^+=\Phi(\beta_{13})\sqcup\Phi(\beta_{23})\sqcup\Phi(\beta_{33})=\Phi(\beta_{33})$ and $\Delta_2^+=\Phi(\beta_{14})\sqcup\Phi(\beta_{24})\sqcup\Phi(\beta_{34})=\Phi(\beta_{14})$. This decomposition is not irreducible: it fails condition (ii) for irreducibility since $\beta_{31}\neq I_3$ and $\beta_{33}\neq I_2$; it also fails condition (iii) since $\beta_{14}\neq I_2$ and $\beta_{21}\neq I_3$.

The recursive form of this theorem allows us to inductively solve many problems concerning decompositions. For example, in §6 we exploit this recursiveness to obtain a number of results enumerating various solutions to the main problem. In §8 we use the form to prove a result about the decompositions which yields an algorithm producing all generating rays on a given regular codimension n face of the Littlewood-Richardson cone.

1.5. The problem discussed above has a Lie theoretic background and a natural generalization. We recommend the book by Fulton and Harris, [FH] as a general Lie Theory reference. Let Δ be a root system with corresponding Weyl group $\mathcal{W}$. Fix a splitting $\Delta = \Delta^+ \sqcup \Delta^-$ of Δ into positive and negative roots. For $\alpha \in \mathcal{W}$, the inversion set of α, $\Phi(\alpha)$ is defined by $\Phi(\alpha) := \{v \in \Delta^+ \mid \alpha \cdot v \in \Delta^-\}$. We are concerned with ways to express the positive roots as a disjoint union of inversion sets: $\Delta^+ = \Phi(\alpha_1) \sqcup \Phi(\alpha_2) \sqcup \cdots \sqcup \Phi(\alpha_r)$ where $\alpha_1, \alpha_2, \ldots, \alpha_r \in \mathcal{W}$.

Problem 1.2, which is solved by Theorem 1.11, is the A_{n-1}-case of the more general problem for arbitrary root systems. (Both of the motivating problems also have versions for arbitrary root systems, and their solution is again in terms of decompositions of the positive roots into inversion sets, i.e., the more general problem.) It is natural to attempt to solve the general problem for all root systems. Section §5 is devoted to studying the root systems of types B, C and D. We provide a solution for root systems of types B and C. Root systems of types B and C have isomorphic Weyl groups and so yield identical answers to our questions; nevertheless we consider them separately because this gives us two different ways of looking at the same problem. Root systems of type D are more complicated and we only provide a brief discussion of the difficulties we encountered when attempting to deal with them. The exceptional root systems are also interesting but our methods are unlikely to yield any results. The solution of Problem 1.2 for root systems of type G_2 is elementary: all nontrivial decompositions are of the form $\Phi(\alpha) \sqcup \Phi(J\alpha)$, where $J \in \mathcal{W}(G_2)$ is the longest element of $\mathcal{W}(G_2)$. The root system F_4 is probably easily treated by direct computations (possibly aided by a computer). Root systems of type E, especially E_8, may be too complicated to treat even by computer computations.

In §7 we use Theorem 1.11 to give a solution, in the form of an algorithm, to the following natural variation of Problem 1.2: Given $\alpha \in S_n$, describe all decompositions

$$\Phi(\alpha) = \Phi(\alpha_1) \sqcup \Phi(\alpha_2).$$

1.6. Here is a brief outline of the contents of the paper. In §2 we establish some basic results on inversion sets. In §3 we define restriction maps and use them to establish further results on simple and irreducible permutations and to prove the main theorem. Section §4 discusses symmetric permutations. The results about symmetric permutations are then used in §5 to extend the main theorem to root systems of types B and C. We then turn to applications of the main theorem. In §6 we give some enumerative results deduced from the theorems. In §7 we give an algorithm to decompose a single inversion set. In §8 we use the main theorem to parameterize regular codimension n faces of the Littlewood-Richardson cone. Finally in §A (an appendix) we describe *sign diagrams*, a visual method of displaying inversion sets which has proved useful to us in thinking about these problems.

2. Preliminaries on inversion sets

It is easy to see that an inversion set Φ must satisfy the following two conditions:

(i) If $(i,j),(j,k) \in \Phi$ then $(i,k) \in \Phi$. (*closed condition*)

(ii) If $(i,j),(j,k) \notin \Phi$ then $(i,k) \notin \Phi$. (*co-closed condition*)

Kostant [K] proved the following statement characterizing inversion sets.

Proposition 2.1 ([K, Proposition 5.10])**.** *A set $\Phi \subset \Delta_n^+$ is an inversion set if and only if Φ is both closed and co-closed, i.e., both Φ and its complement $\Delta_n^+ \setminus \Phi$ satisfy the closed condition.*

The following simple result is often useful.

Lemma 2.2. *Every non-empty inversion set $\Phi(\alpha)$ contains at least one simple root.*

Proof. This follows from the fact that if $\alpha(i) < \alpha(i+1)$ for all $i = 1,2,\dots,n-1$ then $\alpha = I_n$. □

Definition 2.3. The graph of a permutation $\alpha \in S_n$ is the set of n lattice points $\{(i,\alpha(i)) \mid i = 1,2,\dots,n\}$ considered as a subset of $[1,n] \times [1,n] \subset \mathbb{R}^2$.

We have already noted that $\Phi(J) = \Delta^+$. The following two lemmas give further indication of the importance of J.

Lemma 2.4. *Let $\alpha \in S_n$. The permutation α is simple if and only if $J\alpha$ is simple.*

Proof. A block of size $t+1$ for the permutation α corresponds to a $t \times t$ closed square in $[1,n] \times [1,n]$ which contains $t+1$ points of the graph of α. Hence α is simple if there does not exist a $t \times t$ closed square in $[1,n] \times [1,n]$ containing $t+1$ points of the graph of α with $2 \leqslant t \leqslant n-1$. If the graph of α satisfies this condition then so does the graph of $J\alpha$, which is obtained from that of α by reflecting in the horizontal line $y = n/2$. Thus α is simple if and only if $J\alpha$ is simple. □

To each element $(i,j) \in \Delta_n^+$ we associate the line segment joining the points $(i,\alpha(i))$ and $(j,\alpha(j))$ on the graph of α. We note that $(i,j) \in \Phi(\alpha)$ if and only if the corresponding line segment has negative slope.

Lemma 2.5. *Let $\alpha \in S_n$. Then $\Delta_n^+ = \Phi(\alpha) \sqcup \Phi(J\alpha)$, or equivalently, $\Phi(J\alpha) = \Delta_n^+ \setminus \Phi(\alpha)$.*

Proof. The graph of $J\alpha$ is obtained from the graph of α by reflecting in the horizontal line $y = n/2$. Using the characterization of $\Phi(\alpha)$ as those positive roots whose corresponding line segment has negative slope completes the proof of the lemma. □

Corollary 2.6. *The element J_m is reducible for $m \geqslant 3$ and irreducible for $m = 2$.*

Proof. By Lemma 2.5 any $\tau \in S_m \setminus \{J_m, I_m\}$ gives a non-trivial decomposition $\Phi(J_m) = \Delta_m^+ = \Phi(\tau) \sqcup \Phi(J_m\tau)$, and the set $S_m \setminus \{J_m, I_m\}$ is non-empty if $m \geqslant 3$. Conversely, $\Phi(J_2) = \{(1,2)\}$ is clearly irreducible. □

Next we discuss some basic properties of decompositions.

As the following proposition shows, the union of an arbitrary collection of inversion sets appearing in a decomposition is again the inversion set of a permutation.

Proposition 2.7. *Suppose $\Delta_n^+ = \Phi(\alpha_1) \sqcup \Phi(\alpha_2) \sqcup \cdots \sqcup \Phi(\alpha_r)$ is a decomposition and let A be any subset of $\{1, 2, \ldots, r\}$. Then there exists $\alpha \in S_n$ such that $\Phi(\alpha) = \sqcup_{a \in A} \Phi(\alpha_a)$.*

Proof. Clearly it suffices to prove the assertion for doubleton sets $A = \{p, q\}$. Thus it suffices to show that $\Phi(\alpha_p) \sqcup \Phi(\alpha_q)$ is both closed and co-closed. For ease of notation, we will assume $A = \{1, 2\}$. First we show that $\Phi(\alpha_1) \sqcup \Phi(\alpha_2)$ is co-closed. Suppose that $(i, j), (j, k) \notin \Phi(\alpha_1) \sqcup \Phi(\alpha_2)$. Then for $b = 1, 2$ we have $(i, k) \notin \Phi(\alpha_b)$ since $\Phi(\alpha_b)$ is co-closed. Thus $(i, k) \notin \Phi(\alpha_1) \sqcup \Phi(\alpha_2)$ which shows $\Phi(\alpha_1) \sqcup \Phi(\alpha_2)$ is co-closed.

To see that $\Phi(\alpha_1) \sqcup \Phi(\alpha_2)$ is closed, suppose that $(i, j), (j, k) \in \Phi(\alpha_1) \sqcup \Phi(\alpha_2)$. Then for $a = 3, 4, \ldots, r$ we have $(i, j), (j, k) \notin \Phi(\alpha_a)$ and thus $(i, k) \notin \Phi(\alpha_a)$ since $\Phi(\alpha_a)$ is co-closed. Hence $(i, k) \notin \sqcup_{a=3}^r \Phi(\alpha_a)$ which implies that $(i, k) \in \Phi(\alpha_1) \sqcup \Phi(\alpha_2)$. This shows that that $\Phi(\alpha_1) \sqcup \Phi(\alpha_2)$ is closed and completes the proof of proposition. □

Note that some hypothesis of the type $\Delta_n^+ = \Phi(\alpha_1) \sqcup \Phi(\alpha_2) \sqcup \cdots \sqcup \Phi(\alpha_r)$ is necessary in the above proposition; arbitrary unions of inversion sets need not be inversion sets. For example, consider $n = 3$, $\alpha_1 = (2, 1, 3)$ and $\alpha_2 = (1, 3, 2)$. Then $\Phi(\alpha_1) = \{(1, 2)\}$, $\Phi(\alpha_2) = \{(2, 3)\}$ and $\Phi(\alpha_1) \sqcup \Phi(\alpha_2)$ is not closed and so is not an inversion set.

Corollary 2.8. *If $\Phi(\alpha) = \Phi(\alpha_1) \sqcup \Phi(\alpha_2) \sqcup \cdots \sqcup \Phi(\alpha_r)$ and A is any subset of $\{1, 2, \ldots, r\}$ then there exists $\alpha_A \in S_n$ with $\Phi(\alpha_A) = \sqcup_{a \in A} \Phi(\alpha_a)$.*

Proof. Set $\alpha_{r+1} = J\alpha$. By Lemma 2.5 we then have the decomposition $\Delta_n^+ = \Phi(\alpha_1) \sqcup \cdots \sqcup \Phi(\alpha_{r+1})$. The corollary then follows from Proposition 2.7 applied to this decomposition. □

Recall that an element $\alpha \in S_n$ is called reducible if there exists a non-trivial decomposition $\Phi(\alpha) = \Phi(\alpha_1) \sqcup \cdots \sqcup \Phi(\alpha_r)$. By Corollary 2.8 if α is reducible there exists such a non-trivial decomposition with $r = 2$.

Given a decomposition $\Delta_n^+ = \Phi(\alpha_1) \sqcup \cdots \sqcup \Phi(\alpha_r)$ we may obtain finer and coarser decompositions as follows. We obtain a coarser decomposition by choosing one or more disjoint subsets $A_i \subset \{1, 2, \ldots, r\}$ and replacing $\sqcup_{a \in A_i} \Phi(\alpha_a)$ by the single inversion set $\Phi(\alpha_{A_i})$ where α_{A_i} is the element whose existence is guaranteed by Corollary 2.8. We obtain a finer decomposition by further decomposing one or more of the $\Phi(\alpha_i)$ into smaller inversion sets.

We may continue refining a decomposition until we arrive at an *irreducible decomposition*, i.e., a decomposition $\Delta_n^+ = \Phi(\gamma_1) \sqcup \Phi(\gamma_2) \sqcup \cdots \sqcup \Phi(\gamma_s)$ where each $\Phi(\gamma_i)$ is irreducible.

Clearly every decomposition of Δ_n^+ may be obtained by applying the coarsening operation to some irreducible decomposition. For this reason, studying and classifying the irreducible decompositions is of particular interest. Accordingly we characterize the irreducible decompositions in our main theorem, Theorem 1.11, as well as in the analogous theorems for root systems of type B and C.

3. Restriction maps and proof of the main theorem

Given a subset $\mathcal{F} \subseteq \{1, 2, \dots, n\}$ and an element $\alpha \in S_n$ we obtain a permutation in S_m, with $m = |\mathcal{F}|$, by noting how α changes the relative order of elements of $\mathcal{F}$. This procedure gives rise to a map of sets $\theta_{\mathcal{F}} \colon S_n \longrightarrow S_m$ called a *restriction map*. Although not homomorphisms, the maps $\theta_{\mathcal{F}}$ are useful in making inductive arguments on inversion sets. In this section we use restriction maps to establish several results on simple and irreducible permutations, culminating in a proof of the main theorem (Theorem 1.11).

We start by giving formal descriptions of the restriction maps and the inflation procedure.

Definitions 3.1. (a) Two sequences $x_1, x_2, \dots, x_m$ and $y_1, y_2, \dots, y_m$ each comprised of m distinct real numbers are *order isomorphic* if $x_i > x_j$ if and only if $y_i > y_j$.

(b) Suppose $\mathcal{F}$ is a subset of $\{1, 2, \dots, n\}$ of size m, and write $\mathcal{F} = \{i_1, i_2, \dots, i_m\}$ where $i_1 < i_2 < \cdots < i_m$. For any $\alpha \in S_n$ restricting α to $\mathcal{F}$ yields a sequence $\alpha(i_1), \alpha(i_2), \dots, \alpha(i_m)$ which is order isomorphic to the sequence $\mu(1), \mu(2), \dots, \mu(m)$ corresponding to a unique element $\mu \in S_m$. We denote this element μ by $\mu = \theta_{\mathcal{F}}(\alpha)$ and use $\theta_{\mathcal{F}} \colon S_n \longrightarrow S_m$ for the corresponding map of sets.

(c) For $\mathcal{F} \subseteq \{1, 2, \dots, n\}$ we write $\Delta^+_{\mathcal{F}}$ to denote the set $\Delta^+_{\mathcal{F}} := \{(i,j) \in \Delta^+_n \mid i, j \in \mathcal{F}\}$.

(d) A decomposition of $\{1, \dots, n\}$ into an *ordered disjoint union of intervals* is a decomposition $\{1, 2, \dots, n\} = U_1 \sqcup U_2 \sqcup \cdots \sqcup U_m$ where each U_i is an interval and where for each $1 \leqslant i < j \leqslant m$, we have $a < b$ for each $a \in U_i$ and $b \in U_j$.

(e) Given a decomposition of $\{1, \dots, n\}$ into an ordered disjoint union of intervals as above, a subset $\mathcal{F} \subset \{1, 2, \dots, n\}$ is *admissible* if $|\mathcal{F} \cap U_i| = 1$ for all $i = 1, 2, \dots, m$.

Note that the condition of being admissible in (e) depends on the choice of decomposition into ordered disjoint intervals. In every case we use this term we will be careful to make the choice of decomposition explicit.

Suppose now that we are given a decomposition of $\{1, \dots, n\}$ into ordered disjoint intervals $U_1, \dots, U_m$. Choose $\sigma \in S_m$ and $\beta_i \in S_{|U_i|}$ for $i = 1, \dots, m$. In addition to the description by shuffling cards given in §1.3, the inflation $\alpha := \sigma[\beta_1, \beta_2, \dots, \beta_m] \in S_n$, $n = \sum |U_i|$, is characterized by the following two conditions:

(1) $\theta_{U_i}(\alpha) = \beta_i$ for all $i = 1, 2, \dots, m$.
(2) $\theta_{\mathcal{F}}(\alpha) = \sigma$ for any admissible $\mathcal{F}$.

We will frequently use the second fact, which allows us to recover σ using any admissible subset $\mathcal{F}$.

The following lemma, which computes the inversion set of $\alpha = \sigma[\beta_1, \dots, \beta_m]$ from those of its components, follows from either of the descriptions of the inflation procedure. The reader may find an illustration of this lemma and the arguments for its proof in §A.3.

Lemma 3.2. *Suppose $\alpha = \sigma[\beta_1, \beta_2, \dots, \beta_m] \in S_n$ where $U_1, \dots, U_m$ is a decomposition of $\{1, \dots, n\}$ into ordered disjoint intervals, $\sigma \in S_m$, and $\beta_i \in S_{|U_i|}$ for $i = 1, \dots, m$. Let Ψ_i*

denote the order preserving bijection $\Psi_i : U_i \to \{1, 2, \ldots, |U_i|\}$. Then

$$\Phi(\alpha) = \Big\{(a, b) \mid a \in U_i, b \in U_j, (i, j) \in \Phi(\sigma)\Big\} \sqcup \left(\bigsqcup_{i=1}^{m} \Psi_i^{-1}(\Phi(\beta_i))\right).$$

If a permutation σ is reducible it is clear that any inflation $\sigma[I_{z_1}, \ldots, I_{z_m}]$ (for any positive integers $z_1,\ldots, z_m$) is also reducible: one simply takes a decomposition of $\Phi(\sigma)$ and inflates the permutations which appear. However, it is not immediately clear that an inflation of an irreducible element remains irreducible; *a priori* it seems that there could be decompositions of the inflation which do not respect the inflation structure, and therefore do not come from decompositions of the original σ. That this can never happen is a consequence of the following more precise statement.

Lemma 3.3. *For any $\sigma \in S_m$, and any positive integers $z_1,\ldots, z_m$, inflation of the decompositions of $\Phi(\sigma)$ gives a one-to-one correspondence between the decompositions of $\Phi(\sigma)$ and the decompositions of $\Phi(\sigma[I_{z_1}, \ldots, I_{z_m}])$.*

Proof. Set $\alpha = \sigma[I_{z_1}, \ldots, I_{z_m}]$ and $n = \sum z_i$. We must show that for any decomposition $\Phi(\alpha) = \Phi(\alpha_1) \sqcup \cdots \sqcup \Phi(\alpha_r)$ there are unique $\sigma_1,\ldots, \sigma_r \in S_m$ such that $\alpha_k = \sigma_k[I_{z_1}, \ldots, I_{z_m}]$ for $k = 1,\ldots, r$. Lemma 3.2 then implies that $\Phi(\sigma) = \Phi(\sigma_1) \sqcup \cdots \sqcup \Phi(\sigma_r)$.

Let $\{1, 2, \ldots, n\} = U_1 \sqcup U_2 \sqcup \cdots \sqcup U_m$ be the decomposition into ordered disjoint intervals corresponding to the inflation $\sigma[I_{z_1}, I_{z_2}, \ldots, I_{z_m}]$. By Lemma 3.2, we have $\Phi(\alpha) = \{(a, b) \in \Delta_n^+ \mid a \in U_i, b \in U_j, (i, j) \in \Phi(\sigma)\}$.

Choose any root $(i, j) \in \Phi(\sigma)$. The fact that no root (a, a') with $a, a' \in U_i$ is in $\Phi(\alpha)$ means that no such root is in $\Phi(\alpha_1),\ldots, \Phi(\alpha_r)$, and so each of $\alpha_1,\ldots, \alpha_r$ preserves the relative order of the elements in U_i. Similarly each of $\alpha_1, \ldots, \alpha_r$ preserves the relative order of elements in U_j.

Let a_0 be the smallest element in U_i and b_1 the largest element in U_j. The root (a_0, b_1) is in $\Phi(\alpha)$ and so must be contained in one of $\Phi(\alpha_1),\ldots, \Phi(\alpha_r)$. Suppose that $(a_0, b_1) \in \Phi(\alpha_k)$, i.e., that $\alpha_k(b_1) < \alpha_k(a_0)$. Then the fact that α_k preserves the relative order of the elements in U_i and U_j now implies that $U_i \times U_j := \{(a, b) \mid a \in U_i, b \in U_j\} \subseteq \Phi(\alpha_k)$. Since the decomposition of $\Phi(\alpha)$ is into disjoint subsets, we therefore have $(U_i \times U_j) \bigcap \Phi(\alpha_\ell) = \emptyset$ if $\ell \neq k$.

For each $k = 1,\ldots, r$ set $T_k = \{(i, j) \in \Phi(\sigma) \mid U_i \times U_j \subseteq \Phi(\alpha_k)\}$. We have just shown that for any $(i, j) \in \Phi(\sigma)$ there is a unique k such that $(U_i \times U_j) \bigcap \Phi(\alpha_k) \neq \emptyset$, and for that k we have $U_i \times U_j \subseteq \Phi(\alpha_k)$. From this we conclude first that $\Phi(\sigma) = T_1 \sqcup \cdots \sqcup T_r$, and second, since $\Phi(\alpha) = \bigcup_{(i,j)\in\Phi(\sigma)} U_i \times U_j$, that $\Phi(\alpha_k) = \bigcup_{(i,j)\in T_k} U_i \times U_j$ for each k.

The fact that $\Phi(\alpha_k)$ is both closed and co-closed implies that the same holds for T_k, and thus there is a unique permuation $\sigma_k \in S_m$ such that $T_k = \Phi(\sigma_k)$. Lemma 3.2 then says that $\Phi(\sigma_k[I_{z_1}, \ldots, I_{z_k}]) = \bigcup_{(i,j)\in T_k} U_i \times U_j$. Since the inversion set uniquely determines the permutation, we therefore have $\alpha_k = \sigma_k[I_{z_1}, \ldots, I_{z_k}]$ for each $k = 1,\ldots, r$. □

Corollary 3.4. *Let $\sigma \in S_m$, and let $z_1, z_2, \ldots, z_m$ be positive integers. Then the permutation $\alpha := \sigma[I_{z_1}, I_{z_2}, \ldots, I_{z_m}] \in S_n$ is irreducible if and only if σ is irreducible.*

Corollary 3.5. *Let $\alpha = \sigma[\beta_1, \beta_2, \dots, \beta_m] \neq I$ where $\beta_i \in S_{|U_i|}$ for $i = 1, 2, \dots, m$. Then α is irreducible if and only if exactly one of the permutations $\sigma, \beta_1, \beta_2, \dots, \beta_m$ is a non-identity permutation and that non-identity permutation is itself irreducible. In particular, if α is irreducible with $\sigma \neq I$ then $\alpha = \sigma[I, I, \dots, I]$ where σ is irreducible.*

Proof. If $\alpha = \sigma[\beta_1, \dots, \beta_m]$ then it follows immediately from Lemma 3.2 that

$$\Phi(\alpha) = \Phi(\sigma[I_{z_1}, \dots, I_{z_m}]) \sqcup \left(\bigsqcup_{i=1}^{m} \Phi(I_m[I_{z_1}, \dots, I_{z_{i-1}}, \beta_i, I_{z_{i+1}}, \dots, I_{z_m}]) \right),$$

where $z_i = |U_i|$ for $i = 1, \dots, m$. If α is irreducible all then all but one of the inversion sets in the decomposition on the right are empty, and hence all but one of the corresponding elements are the identity. Conversely, if more than one of the inversion sets in this decomposition of $\Phi(\alpha)$ is non-empty then we have a non-trivial decomposition of α. Furthermore, if σ is reducible then Lemma 3.3 shows that α is reducible too. Similarly if some β_i is reducible then the order preserving bijection Φ_i from Lemma 3.2 induces a decomposition of $\Phi(I_m[I_{z_1}, \dots, I_{z_{i-1}}, \beta_i, I_{z_{i+1}}, \dots, I_{z_m}])$, showing again that α is reducible. □

Definition 3.6. Let $\alpha \in S_n$, and $\mathcal{F}, \mathcal{F}' \subset \{1, 2, \dots, n\}$. We say that $\mathcal{F}$ and $\mathcal{F}'$ are *α-connected* if

(i) $\theta_{\mathcal{F}}(\alpha)$ and $\theta_{\mathcal{F}'}(\alpha)$ are irreducible;
(ii) $\Phi(\alpha) \cap \Delta^+_{\mathcal{F}} \cap \Delta^+_{\mathcal{F}'} \neq \emptyset$.

The following two results will be used several times.

Lemma 3.7. *Let $\alpha \in S_n$ and $\mathcal{F}, \mathcal{F}' \subset \{1, 2, \dots, n\}$. Assume that $\mu = \theta_{\mathcal{F}}(\alpha) \neq I$ and $\mu' = \theta_{\mathcal{F}'}(\alpha) \neq I$. Suppose that $\Phi(\alpha) = \Phi(\alpha_1) \sqcup \Phi(\alpha_2) \sqcup \cdots \sqcup \Phi(\alpha_r)$.*

1. *If μ is irreducible then there exists a unique index $\delta(\mathcal{F})$ with $1 \leqslant \delta(\mathcal{F}) \leqslant r$ such that $\Phi(\alpha) \cap \Delta^+_{\mathcal{F}} \subseteq \Phi(\alpha_{\delta(\mathcal{F})})$, and hence $\Phi(\alpha) \cap \Delta^+_{\mathcal{F}} \cap \Phi(\alpha_i) = \emptyset$ for all $i \neq \delta(\mathcal{F})$.*
2. *If $\mathcal{F}$ and $\mathcal{F}'$ are α-connected then $\delta(\mathcal{F}) = \delta(\mathcal{F}')$.*

Proof. Let $m = |\mathcal{F}|$ and suppose that μ is irreducible. Put $\mu_a = \theta_{\mathcal{F}}(\alpha_a)$ for $a = 1, 2, \dots, r$. There exists an order preserving bijection $\Psi : \mathcal{F} \to \{1, 2, \dots, m\}$. It is easy to see that $(i, j) \in \Phi(\alpha) \cap \Delta^+_{\mathcal{F}}$ if and only if $(\Psi(i), \Psi(j)) \in \Phi(\theta_{\mathcal{F}}(\alpha))$. Thus Ψ identifies $\Phi(\alpha) \cap \Delta^+_{\mathcal{F}}$ with $\Phi(\theta_{\mathcal{F}}(\alpha))$. Intersecting $\Phi(\alpha) = \Phi(\alpha_1) \sqcup \Phi(\alpha_2) \sqcup \cdots \sqcup \Phi(\alpha_r)$ with $\Delta^+_{\mathcal{F}}$ and using this identification we get $\Phi(\mu) = \Phi(\mu_1) \sqcup \Phi(\mu_2) \sqcup \cdots \sqcup \Phi(\mu_r)$. Since μ is irreducible, there exists a unique $\delta(\mathcal{F})$ such that $\Phi(\mu) = \Phi(\mu_{\delta(\mathcal{F})})$, furthermore $\Phi(\mu_i) = \emptyset$ for all $i \neq \delta(\mathcal{F})$. Therefore $\Phi(\alpha) \cap \Delta^+_{\mathcal{F}} \subseteq \Phi(\alpha_{\delta(\mathcal{F})})$ and $\Phi(\alpha) \cap \Delta^+_{\mathcal{F}} \cap \Phi(\alpha_i) = \emptyset$ for all $i \neq \delta(\mathcal{F})$.

For the second assertion, recall that μ and μ' are irreducible by definition. By the above, $\Phi(\alpha) \cap \Delta^+_{\mathcal{F}} \subseteq \Phi(\alpha_{\delta(\mathcal{F})})$ and $\Phi(\alpha) \cap \Delta^+_{\mathcal{F}'} \subseteq \Phi(\alpha_{\delta(\mathcal{F}')})$. Since $\mathcal{F}$ and $\mathcal{F}'$ are α-connected there exists $(i, j) \in \Phi(\alpha) \cap \Delta^+_{\mathcal{F}} \cap \Delta^+_{\mathcal{F}'}$. Thus $(i, j) \in \Phi(\alpha_{\delta(\mathcal{F})}) \cap \Phi(\alpha_{\delta(\mathcal{F}')})$. Hence $\delta(\mathcal{F}) = \delta(\mathcal{F}')$. □

Corollary 3.8. *Suppose $\mathcal{F}_1, \mathcal{F}_2, \dots, \mathcal{F}_s \subset \{1, 2, \dots, n\}$ where $\mathcal{F}_i$ and $\mathcal{F}_{i+1}$ are α-connected for all $1 \leqslant i \leqslant s - 1$. (In particular, $\theta_{\mathcal{F}_i}(\alpha)$ is irreducible for all $i = 1, 2, \dots, s$.) Assume further that $\Phi(\alpha) \subseteq \cup_{i=1}^{s} \Delta^+_{\mathcal{F}_i}$. Then α is irreducible.*

Proof. Suppose that $\Phi(\alpha) = \Phi(\alpha_1) \sqcup \Phi(\alpha_2) \sqcup \cdots \sqcup \Phi(\alpha_r)$. By Lemma 3.7, we have $j = \delta(\mathcal{F}_1) = \delta(\mathcal{F}_2) = \cdots = \delta(\mathcal{F}_r)$ with $\Phi(\alpha) \cap \Delta^+_{\mathcal{F}_i} \subseteq \Phi(\alpha_j)$. Therefore $\Phi(\alpha) = \Phi(\alpha) \cap (\cup_{i=1}^s \Delta^+_{\mathcal{F}_i}) = \cup_{i=1}^s (\Phi(\alpha) \cap \Delta^+_{\mathcal{F}_i}) \subset \Phi(\alpha_j)$ and thus the decomposition $\Phi(\alpha) = \Phi(\alpha_1) \sqcup \Phi(\alpha_2) \sqcup \cdots \sqcup \Phi(\alpha_r)$ is trivial. □

The basic objects for describing a permutation by inflation are the simple permutations, while in describing decompositions the basic objects are the irreducible permutations. In our recursive method of describing decompositions by inflations it is therefore natural to choose the basic object to be those permutations which are both simple and irreducible.

It turns out that simple permutations are automatically irreducible (and thus our basic building blocks are again the simple permutations). To prove this and a related useful fact we need an additional definition. Recall that a permuation $\alpha \in S_n$ is simple if it has no blocks of length t for $2 \leqslant t \leqslant n-1$.

Definition 3.9. A permutation $\alpha \in S_n$ is *two-block simple* if $\alpha(1) \neq 1$, $\alpha(i+1) \neq \alpha(i)+1$ for $1 \leqslant i \leqslant n-1$, and $\alpha(n) \neq n$.

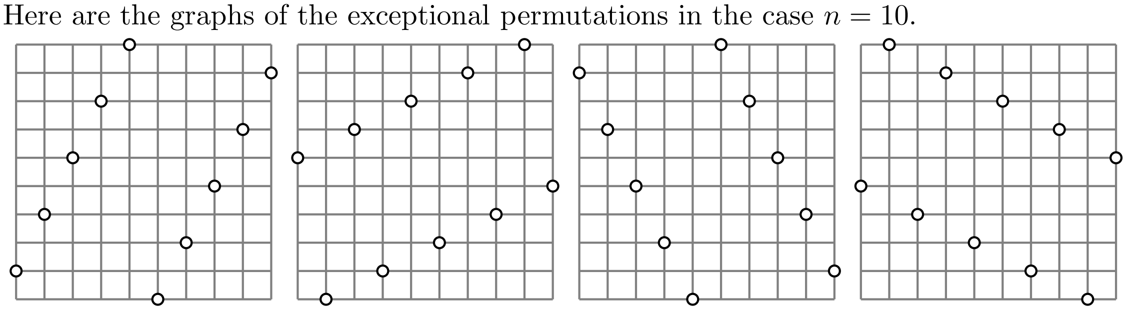
The name is somewhat inaccurate: the condition that α has no blocks of length two is that $\alpha(i+1) \neq \alpha(i) \pm 1$ for all i, whereas we are only asking that $\alpha(i+1) \neq \alpha(i)+1$ for all i, and imposing the additional conditions that $\alpha(1) \neq 1$ and $\alpha(n) \neq n$. Nonetheless, we continue to use this name since it gives an indication of the defining conditions. In Proposition 3.15 below we will show that the property of being simple is equivalent to the property of being both irreducible and two-block simple. From this equivalence and Lemma 2.4 we will deduce that if α is simple then $J\alpha$ is irreducible.

Our method of proving the equivalence is inductive. The base cases of the induction are a particular family of permutations previously appearing in the literature.

Definition 3.10. ([AA, Definition 4]) Let $n = 2m$ be even with $m \geqslant 2$. A permutation $\alpha \in S_n$ is *exceptional* if it is one of the following permutations

(1) $\alpha = (2, 4, 6, \ldots, 2m-2, 2m, 1, 3, 5, \ldots, 2m-3, 2m-1)$,
(2) $\alpha = (m+1, 1, m+2, 2, m+3, 3, \ldots, 2m-1, m-1, 2m, m)$,
(3) $\alpha = (2m-1, 2m-3, 2m-5, \ldots, 3, 1, 2m, 2m-2, 2m-4, \ldots, 4, 2)$,
(4) $\alpha = (m, 2m, m-1, 2m-1, m-2, 2m-2, \ldots, 2, m+2, 1, m+1)$.

Here are the graphs of the exceptional permutations in the case $n = 10$.



(1) (2) (3) (4)

These patterns, which hold for all even n, allow for a quick verification of the fact that the exceptional permutations are "simple and do not have simple one point deletions". Moreover this property characterizes exceptional permutations. (See Definition 3.12 and Theorem 3.13 below.)

Lemma 3.11. *Let $\alpha \in S_n$ be exceptional. Then α is irreducible and two-block simple.*

Proof. It is easily seen that all of these permutations are two-block simple.

(1) Suppose $\alpha = (2, 4, 6, \ldots, 2m-2, 2m, 1, 3, 5, \ldots, 2m-3, 2m-1)$. Then $\Phi(\alpha)$ has only one simple root, $(m, m+1)$, and so is irreducible.

For the remaining cases, we proceed by induction using Corollary 3.8 repeatedly.

(2) Suppose $\alpha = (m+1, 1, m+2, 2, m+3, 3, \ldots, 2m-1, m-1, 2m, m)$. For $m = 2$ we check directly that $\alpha = (3, 1, 4, 2)$ is irreducible. Let $m \geqslant 3$ and set

$$\begin{array}{lllll} \mathcal{F}_1 & := & \{1, 2, \ldots, 2m-2\} & = & \{1, 2, \ldots, 2m\} \backslash \{2m-1, 2m\} \\ \mathcal{F}_2 & := & \{3, 4, \ldots, 2m\} & = & \{1, 2, \ldots, 2m\} \backslash \{1, 2\} \\ \mathcal{F}_3 & := & \{1, 2, 2m-1, 2m\}. & & \end{array}$$

Then $\theta_{\mathcal{F}_1}(\alpha) = \theta_{\mathcal{F}_2}(\alpha) = (m, 1, m+1, 2, m+2, 3, \ldots, 2m-3, m-2, 2m-2, m-1)$ is irreducible by the induction assumption and $\theta_{\mathcal{F}_3}(\alpha) = (3, 1, 4, 2)$ is irreducible by the base case. Furthermore, $(3, 4) \in \Phi(\alpha) \cap \Delta^+_{\mathcal{F}_1} \cap \Delta^+_{\mathcal{F}_2}$ and $(2m-1, 2m) \in \Phi(\alpha) \cap \Delta^+_{\mathcal{F}_2} \cap \Delta^+_{\mathcal{F}_3}$ together with the observation that $\Phi(\alpha) \subset \Delta^+ = \Delta^+_{\mathcal{F}_1} \cup \Delta^+_{\mathcal{F}_2} \cup \Delta^+_{\mathcal{F}_3}$ imply that Corollary 3.8 applies and hence α is irreducible.

(3) Suppose $\alpha = (2m-1, 2m-3, 2m-5, \ldots, 3, 1, 2m, 2m-2, 2m-4, \ldots, 4, 2)$. For $m = 2$, $\alpha = (3, 1, 4, 2)$, as discussed above, is irreducible. It is not too difficult to check directly that, for $m = 3$, $\alpha = (5, 3, 1, 6, 4, 2)$ is irreducible as well. Let $m \geqslant 4$ and set

$$\begin{array}{lllll} \mathcal{F}_1 & := & \{1, 2, \ldots, m-1, m+1, \ldots, 2m-1\} & = & \{1, 2, \ldots, 2m\} \backslash \{m, 2m\} \\ \mathcal{F}_2 & := & \{2, 3, \ldots, m, m+2, \ldots, 2m\} & = & \{1, 2, \ldots, 2m\} \backslash \{1, m+1\} \\ \mathcal{F}_3 & := & \{2, m, m+1, 2m\} & & \\ \mathcal{F}_4 & := & \{1, m, m+1, 2m\}. & & \end{array}$$

Then $\theta_{\mathcal{F}_1}(\alpha) = \theta_{\mathcal{F}_2}(\alpha) = (2m-3, 2m-5, \ldots, 3, 1, 2m-2, 2m-4, \ldots, 4, 2)$ is irreducible by the induction assumption and $\theta_{\mathcal{F}_3}(\alpha) = \theta_{\mathcal{F}_4}(\alpha) = (3, 1, 4, 2)$ is irreducible by the base case. Furthermore, $(2, m+3) \in \Phi(\alpha) \cap \Delta^+_{\mathcal{F}_1} \cap \Delta^+_{\mathcal{F}_2}$, $(2, 2m) \in \Phi(\alpha) \cap \Delta^+_{\mathcal{F}_2} \cap \Delta^+_{\mathcal{F}_3}$, and $(m+1, 2m) \in \Phi(\alpha) \cap \Delta^+_{\mathcal{F}_3} \cap \Delta^+_{\mathcal{F}_4}$ together with the observation that $\Phi(\alpha) \subset \Delta^+ = \Delta^+_{\mathcal{F}_1} \cup \Delta^+_{\mathcal{F}_2} \cup \Delta^+_{\mathcal{F}_3} \cup \Delta^+_{\mathcal{F}_4}$ imply that Corollary 3.8 applies and hence α is irreducible.

(4) Finally suppose that $\alpha = (m, 2m, m-1, 2m-1, m-2, 2m-2, \ldots, 2, m+2, 1, m+1)$. For $m = 2$ we check directly that $\alpha = (2, 4, 1, 3)$ is irreducible. It is not too difficult to check directly that, for $m = 3$, $\alpha = (3, 6, 2, 5, 1, 4)$ is irreducible as well. Let $m \geqslant 4$ and set

$$\begin{array}{lllll} \mathcal{F}_1 & := & \{1, 2, \ldots, 2m-3, 2m-2\} & = & \{1, 2, \ldots, 2m\} \backslash \{2m-1, 2m\} \\ \mathcal{F}_2 & := & \{1, 2, \ldots, 2m-3, 2m\} & = & \{1, 2, \ldots, 2m\} \backslash \{2m-2, 2m-1\} \\ \mathcal{F}_3 & := & \{3, 4, \ldots, 2m-1, 2m\} & = & \{1, 2, \ldots, 2m\} \backslash \{1, 2\}. \end{array}$$

Then $\theta_{\mathcal{F}_1}(\alpha) = \theta_{\mathcal{F}_2}(\alpha) = \theta_{\mathcal{F}_3}(\alpha) = (m-1, 2m-2, m-2, 2m-3, \dots, 2, m+1, 1, m)$ is irreducible by the induction assumption. Furthermore, $(2,3) \in \Phi(\alpha) \cap \Delta^+_{\mathcal{F}_1} \cap \Delta^+_{\mathcal{F}_2}$ and $(3, 2m-3) \in \Phi(\alpha) \cap \Delta^+_{\mathcal{F}_2} \cap \Delta^+_{\mathcal{F}_3}$, together with the observation that $\Phi(\alpha) \subset \Delta^+ = \Delta^+_{\mathcal{F}_1} \cup \Delta^+_{\mathcal{F}_2} \cup \Delta^+_{\mathcal{F}_3}$ imply that Corollary 3.8 applies and hence α is irreducible. □

We now turn to the reduction step and then the inductive proof of Proposition 3.15.

Definition 3.12. Let $\alpha \in S_n$. Choose k with $1 \leqslant k \leqslant n$ and put $\mathcal{F} = \{1, 2, \dots, n\} \setminus \{k\}$. The permutation $\alpha^\circ = \theta_{\mathcal{F}}(\alpha) \in S_{n-1}$ is called a *one point deletion* of α. We say that α° is obtained from α by deleting $(k, \alpha(k))$.

The following theorem, expressed in the language of posets, was first proved by Schmerl and Trotter [ST]. The version below in terms of permutations appears as [AA, Theorem 5].

Theorem 3.13. *Let $n \geqslant 2$ and suppose $\alpha \in S_n$ is simple but not exceptional. Then α has a one point deletion α° which is simple.*

Lemma 3.14. *Suppose that $\alpha \in S_n$ is reducible and has a one-point deletion which is irreducible. Then α is not simple.*

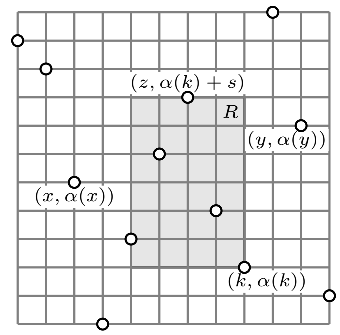
Proof. Let $\Phi(\alpha) = \Phi(\alpha_1) \sqcup \Phi(\alpha_2)$ be a non-trivial decomposition, and let α° be an irreducible one-point deletion obtained from α by deleting $(k, \alpha(k))$. Since α° is irreducible, applying Lemma 3.7(1) with $\mathcal{F} = \{1, \dots, n\} \setminus \{k\}$ gives that either $\Phi(\alpha) \cap \Delta^+_{\mathcal{F}} \subset \Phi(\alpha_1)$ or $\Phi(\alpha) \cap \Delta^+_{\mathcal{F}} \subset \Phi(\alpha_2)$. By relabeling we may assume that $\Phi(\alpha) \cap \Delta^+_{\mathcal{F}} \subset \Phi(\alpha_1)$. Concretely this means that all roots of the form $(i,j) \in \Phi(\alpha)$ with either $i \neq k$ or $j \neq k$ are in $\Phi(\alpha_1)$. The remaining roots in $\Phi(\alpha)$, those of the form (i,k) or (k,j) may appear in either $\Phi(\alpha_1)$ or $\Phi(\alpha_2)$, and $\Phi(\alpha_2)$ only has roots of this form. Furthermore, since the decomposition is assumed non-trivial, there is at least one root in $\Phi(\alpha_2)$.

Write $|\Phi(\alpha_2)| = p + q$ where p of the elements of $\Phi(\alpha_2)$ are of the form (i,k) with $1 \leqslant i < k$ and q of the elements of $\Phi(\alpha_2)$ are of the form (k,j) with $k < j \leqslant n$. Suppose both p and q are non-zero. Then there exist $i < k$ and $j > k$ with $(i,k), (k,j) \in \Phi(\alpha_2)$. Since $\Phi(\alpha_2)$ is closed this means that the root $(i,j) \in \Phi(\alpha_2)$ contrary to the description above. Thus only one of p and q is non-zero.

Assume first that $p \neq 0$ and $q = 0$. By the form of the roots in $\Phi(\alpha_2)$ the only simple root in $\Phi(\alpha_2)$ is $(k-1, k)$. This implies that α_2 preserves the relative order of the elements in $\{1, 2, \dots, k-1\}$ and the relative order of the elements in $\{k, k+1, \dots, n\}$. Along with the fact that the only roots in $\Phi(\alpha_2)$ are of the form (i,k), this implies that $\Phi(\alpha_2) = \{(k-p, k), (k-p+1, k), \dots, (k-1, k)\}$.

Set $s = \max\{\alpha(k-i) \mid 1 \leqslant i \leqslant p\} - \alpha(k)$, and let R be the $p \times s$ rectangle $R := [k-p, k] \times [\alpha(k), \alpha(k) + s]$. There are $p+1$ vertical lattice lines and $s+1$ horizontal lattice lines through R. Exactly $p+1$ points of the graph of α lie inside the rectangle R: $(k-p, \alpha(k-p)), (k-p+1, \alpha(k-p+1)), \dots, (k-1, \alpha(k-1))$ and $(k, \alpha(k))$. Since α is a permutation (and hence injective), no two points of its graph may lie on the same horizontal line, and thus $s \geqslant p$. We will now show that $s \leqslant p$ and hence $s = p$.

We first claim that there are no points of the graph of α strictly to the left of R, i.e., a point $(x, \alpha(x))$ with $x < k - p$ and $\alpha(k) < \alpha(x) < \alpha(k) + s$. Assume to the contrary that $(x, \alpha(x))$ is such a point. A potential graph of such an α is shown below right (although, as part of the proof will show, certain features of the graph are incorrect). Let $\mathcal{F} = \{x, z, k\}$ with $z := \alpha^{-1}(\alpha(k) + s)$. By this choice of x, the slope between $(x, \alpha(x))$ and $(k, \alpha(k))$ is negative, and so (x, k) is a root of $\Phi(\alpha)$. This root is not contained in $\Phi(\alpha_2)$ since if $(i, k) \in \Phi(\alpha_2)$ then $(i, \alpha(i))$ is in R, and we have chosen $(x, \alpha(x))$ outside of R. Thus $(x, k) \in \Phi(\alpha_1)$, and hence $\Phi(\theta_{\mathcal{F}}(\alpha_1)) \neq \emptyset$. On the other hand, the slope between $(z, \alpha(k) + s)$ and $(k, \alpha(k))$ is also negative, and thus $(z, k) \in \Phi(\alpha)$. Since $(z, \alpha(k))$ is in R, this root is in $\Phi(\alpha_2)$, and hence $\Phi(\theta_{\mathcal{F}}(\alpha_2)) \neq \emptyset$. Thus applying $\theta_{\mathcal{F}}$ to the decomposition of $\Phi(\alpha)$ produces a non-trivial decomposition of $\theta_{\mathcal{F}}(\alpha) = (2, 3, 1)$. However the permutation $(2, 3, 1)$ is irreducible, and this contradiction establishes the claim.



We similarly claim that there are no points of the graph of α strictly to the right of R. Assume such a point $(y, \alpha(y))$ exists with $k < y$ and $\alpha(k) < \alpha(y) < \alpha(k) + s$. Set $\mathcal{F} = \{z, k, y\}$ with $z = \alpha^{-1}(\alpha(k) + s)$ as above. As before, we deduce that $(z, y) \in \Phi(\alpha_1)$ and so $\Phi(\theta_{\mathcal{F}}(\alpha_1)) \neq \emptyset$, that $(z, k) \in \Phi(\alpha_2)$ and so $\Phi(\theta_{\mathcal{F}}(\alpha_2)) \neq \emptyset$, and hence that the decomposition of $\Phi(\alpha)$ induces a nontrivial decomposition of the irreducible element $\theta_{\mathcal{F}}(\alpha) = (3, 1, 2)$. Thus there are no points on the graph of α to the right of R either.

Since α is a permutation (and hence surjective) there is a point of its graph on each of the $s + 1$ horizontal lattice lines through R. We have just shown that none of these points lie outside of R, and hence all are in R. Each of these points lies on a different vertical lattice line of which there are $p + 1$, and so $s \leqslant p$. We conclude that $p = s$, that R is a square, and that the graph of α contains $p+1$ points in R. If $p+1 < n$ then α is not simple because it has the block of size $p+1$ corresponding to R. If $p+1 = n$ then $(k, \alpha(k)) = (n, 1)$ and α is not simple because it has the block $\{1, 2, \ldots, n-1\}$ of size $n - 1$.

A similar argument, with a rectangle of the form $R = [k, k+q] \times [\alpha(k), \alpha(k) - s]$, handles the case $p = 0$ and $q \neq 0$. $\square$

We can now prove our characterization of simple permutations.

Proposition 3.15. *Let $n \geqslant 2$ and let $\alpha \in S_n$ with $\alpha \neq I_2$. Then $\alpha \in S_n$ is simple if and only if it is irreducible and two-block simple.*

Proof. First suppose α is irreducible and two-block simple, and express α in simple form: $\alpha = \sigma[\beta_1, \beta_2, \ldots, \beta_m]$ with $\beta_b \in S_{z_b}$ for $b = 1, 2, \ldots, m$. By Corollary 3.5, exactly one of the permutations $\sigma, \beta_1, \beta_2, \ldots, \beta_m$ is a non-identity permutation. Since α is two-block simple we have $\alpha(1) \neq 1$ and $\alpha(n) \neq n$. If $\sigma = I_m$ then this would imply that $\beta_1 \neq I$ and $\beta_m \neq I$, which is a contradiction. Thus $\sigma \neq I$ and $\alpha = \sigma[I_{z_1}, I_{z_2}, \ldots, I_{z_m}]$. Since α is two-block simple, $z_b = 1$ for all b and hence $\alpha = \sigma$. Now either σ is simple or $\sigma = J_m$. In the former case, $\alpha = \sigma$ is simple. Otherwise, if $\alpha = J_m$, then we must have $m = 2$ by Corollary 2.6 and thus $\alpha = J_2$ is simple.

Next we suppose that α is simple. We proceed by induction on n. For $n = 2$, we must have $\alpha = J_2$ which is simple, two-block simple and irreducible. For $n = 3$, no elements are simple. For $n \geqslant 4$, it follows immediately from the definitions that α is two-block simple. It remains for us to prove, by induction, that simple implies irreducible.

For $n = 4$ the only two simple permutations are $(2, 4, 1, 3)$ and $(3, 1, 4, 2)$, both of which are exceptional (they appear as (1) and (2) on the list with $m = 2$) and hence irreducible by Lemma 3.11.

Suppose $n \geqslant 5$ and that $\alpha \in S_n$ is simple. If α is exceptional then the result follows from Lemma 3.11. If not, then by Theorem 3.13, α has a one-point deletion α° which is also simple, and hence irreducible by the inductive hypothesis. But then α must also be irreducible. If not, then Lemma 3.14 would apply to show that α is not simple, contrary to assumption. $\square$

Remark 3.16. The permutation $\alpha = (2, 4, 5, 1, 3) \in S_5$ is irreducible since its inversion set contains only one simple root. However α is not two-block simple (and hence also not simple). I.e., although the condition of being simple implies that of being irreducible, the reverse implication does not hold.

Corollary 3.17. *Suppose that $\alpha \in S_n$ is simple. Then $J\alpha$ is irreducible.*

Proof. By Lemma 2.4 the fact that α is simple implies that $J\alpha$ is simple. But then $J\alpha$ is irreducible (and also two-block simple) by Proposition 3.15. $\square$

This corollary is required for the proof of the main theorem and was one of the motivations for proving Proposition 3.15.

We will also need the following lemma in the proof of the main theorerm.

Lemma 3.18. *Let $\alpha \in S_n$ be simple with $n \geqslant 4$. Fix k with $1 \leqslant k \leqslant n$. Then there exists $(i, j) \in \Phi(\alpha)$ with $i \neq k$ and $j \neq k$.*

Proof. Define $a := \alpha^{-1}(1)$ and $b := \alpha^{-1}(n)$. Then $a \neq 1$, $a \neq n$, $b \neq 1$ and $b \neq n$ since α is simple. Clearly $(1, a), (b, n) \in \Phi(\alpha)$. The conditions $1 \neq n$, $a \neq b$, $b \neq 1$ and $a \neq n$ imply that $\{1, a\} \cap \{b, n\} = \emptyset$. Thus either $k \notin \{1, a\}$ or $k \notin \{b, n\}$. Hence one of $(1, a)$ or (b, n) may be used as the required root (i, j). $\square$

We now prove the main theorem.

Proof of Theorem 1.11. Suppose that $\Delta_n^+ = \Phi(\alpha_1) \sqcup \Phi(\alpha_2) \sqcup \cdots \sqcup \Phi(\alpha_r)$ is a decomposition with $\Phi(\alpha_s) \neq \emptyset$ for all s, and express $\alpha_1 = \sigma_1[\beta_{11}, \beta_{12}, \ldots, \beta_{1m}]$ in simple form. Recall that by assumption the highest root $(1, n)$ is an element of $\Phi(\alpha_1)$. Let $\{1, 2, \ldots, n\} = U_1 \sqcup U_2 \sqcup \ldots \sqcup U_m$ be the intervals corresponding to the simple form $\alpha_1 = \sigma_1[\beta_{11}, \beta_{12}, \ldots, \beta_{1m}]$ with $z_b = |U_b|$. Throughout this proof "admissible" refers to the intervals $U_1, \ldots, U_m$.

The assumption that $(1, n) \in \Phi(\alpha_1)$ means that $\sigma_1 \neq I_m$. Corollary 1.9 then implies that either $\sigma_1 \in S_m$ is simple with $m \geqslant 4$ or that $\sigma_1 = J_m$ (These two possibilities correspond to the cases that α_1 is minus-indecomposible or is minus-decomposible, respectively.)

We first consider the case that $\sigma_1 \in S_m$ is simple with $m \geqslant 4$. We next show that there is an $s \geqslant 2$ such that α_s is of the form $\alpha_s = (J_m\sigma_1)[\beta_{s1}, \beta_{s2}, \dots, \beta_{sm}]$. Let $\mathcal{F}$ be an admissible set. Then $\theta_{\mathcal{F}}(\alpha_1) = \sigma_1$ and $\Phi(\sigma_1) \sqcup \Phi(\theta_{\mathcal{F}}(\alpha_2)) \sqcup \cdots \sqcup \Phi(\theta_{\mathcal{F}}(\alpha_r)) = \Delta^+_{\mathcal{F}}$. Thus $\Phi(J_m\sigma_1) = \Phi(\theta_{\mathcal{F}}(\alpha_2)) \sqcup \Phi(\theta_{\mathcal{F}}(\alpha_3)) \sqcup \cdots \sqcup \Phi(\theta_{\mathcal{F}}(\alpha_r))$. The element $J\sigma_1$ is irreducible by Corollary 3.17 and thus there exists $\delta(\mathcal{F}) \geqslant 2$ such that $\Phi(J\sigma_1) = \Phi(\theta_{\mathcal{F}}(\alpha_{\delta(\mathcal{F})}))$, and $\Phi(\theta_{\mathcal{F}}(\alpha_s)) = \emptyset$ for all $s \neq \delta(\mathcal{F}), s \geqslant 2$. We claim that the number $\delta(\mathcal{F})$ is independent of the choice of admissible set $\mathcal{F}$.

Recall that an admissible set is the choice of a single element from each of the sets U_1, U_2, …, U_m, and thus the admissible sets are in one to one correspondence with the points of $U_1 \times U_2 \times \cdots \times U_m$. Given any two admissible subsets $\mathcal{F}$ and $\mathcal{F}'$ we may find a sequence of admissible subsets $\mathcal{F} = \mathcal{F}_0$, $\mathcal{F}_1$, …, $\mathcal{F}_{l-1}$, $\mathcal{F}_l = \mathcal{F}'$ such that each $\mathcal{F}_i$ and $\mathcal{F}_{i+1}$ differ by only a single element (i.e, under the correspondence with elements of $U_1 \times \cdots \times U_m$, differ in only a single coordinate). To prove that $\delta(\mathcal{F})$ is independent of the choice of admissible set we may thus reduce to the case that $\mathcal{F}$ and $\mathcal{F}'$ differ by a single element.

Suppose that $\mathcal{F}'$ is obtained from $\mathcal{F}$ by replacing $u_k \in U_k$ with $u'_k \in U_k$ for some $1 \leqslant k \leqslant m$. There is a root (i, j) in $\Phi(J\sigma_1)$ with $i \neq k$ and $j \neq k$ by Lemma 3.18. Since $\mathcal{F}$ and $\mathcal{F}'$ differ only in the element in U_k, the elements they choose from U_i (respectively U_j) are the same. Set $a = \Delta_{\mathcal{F}} \cap U_i = \Delta_{\mathcal{F}'} \cap U_i$ and $b = \Delta_{\mathcal{F}} \cap U_j = \Delta_{\mathcal{F}'} \cap U_j$. Then $(i, j) \in \Phi(\theta_{\mathcal{F}}(\alpha_s))$ if and only if $(a, b) \in \Phi(\alpha_s)$, and similarly $(i, j) \in \Phi(\theta_{F'}(\alpha_s))$ if and only if $(a, b) \in \Phi(\alpha_s)$. We have seen above that $\Phi(\theta_{\mathcal{F}}(\alpha_s)) = \emptyset$ (respectively $\Phi(\theta_{\mathcal{F}'}(\alpha_s)) = \emptyset$) for all $s \neq \delta(\mathcal{F})$ (respectively, $s \neq \delta(\mathcal{F}')$), $s \geqslant 2$. Thus $\delta(\mathcal{F}) = \delta(\mathcal{F}')$, and so $\delta(\mathcal{F})$ is constant for all admissible sets $\mathcal{F}$. By reordering the elements $\alpha_2, \dots, \alpha_r$, we may assume that this constant value is 2.

The statement we have just proved, that $\Phi(\theta_{\mathcal{F}}(\alpha_2)) = \Phi(J\sigma_1)$ for all admissible sets $\mathcal{F}$, is equivalent to the statement that for any $(i, j) \in \Delta_m$, and any $a \in U_i$, $b \in U_j$, $(a, b) \in \Phi(\alpha_2)$ if and only if $(i, j) \in \Phi(J\sigma_1)$. This implies that α_2 permutes the intervals $U_1,\dots, U_m$, in the manner specified by $J\sigma_1$ and thus can be written as an inflation $\alpha_2 = (J\sigma_1)[\beta_{21}, \dots, \beta_{2,m}]$. Specifically, $\beta_{2t} = \theta_{U_t}(\alpha_2)$ for $t = 1,\dots, m$.

For the remaining α_s, $s = 3,\dots, r$, we have $\alpha_s(a) < \alpha_s(b)$ for all $a \in U_i$, $b \in U_j$ and $1 \leqslant i < j \leqslant m$ since the roots (a, b) are all contained in $\Phi(\alpha_1) \sqcup \Phi(\alpha_2)$. This implies that for each such α_s we have $\alpha_s(U_i) = U_i$ for each i and hence that $\alpha_s = I_m[\beta_{s1}, \dots, \beta_{sm}]$ with $\beta_{si} = \theta_{U_i}(\alpha_s)$ for $i = 1,\dots, m$.

On the other hand, if α_1 is minus-decomposable then (by definition) the simple form of α_1 is $\alpha_1 = J_m[\beta_{11}, \beta_{12}, \dots, \beta_{1m}]$. For the remaining s, $s = 2,\dots, r$, we again have that $\alpha_s(a) < \alpha_s(b)$ whenever $a \in U_i$, $b \in U_j$, and $1 \leqslant i < j \leqslant m$ since the roots (a, b) are all contained in $\Phi(\alpha_1)$. We conclude as in the first case that each α_s, $s \geqslant 2$ is of the form $I_m[\beta_{s1}, \dots, \beta_{sm}]$ with $\beta_{si} = \theta_{U_i}(\alpha_s)$ for $i = 1,\dots, m$.

This proves the theorem in the case of a general (possibly reducible) decomposition.

Next we consider irreducible decompositions. Corollary 3.5, shows the necessity of conditions (i), (ii) and (iii) of the final assertion. The element J_m is irreducible if and only if $m = 2$ and this shows the necessity of condition (iv). Thus these four conditions hold for irreducible decompositions. Finally if these four conditions hold it is clear that each of the inversion sets $\Phi(\alpha_a)$ is irreducible by Corollary 3.5. □

4. Symmetric permutations

The aim of this section is to extend the results obtained so far to a special class of permutations. The results will then be used in the next section to study root systems of types B, C and D.

A permutation $\alpha \in S_N$ is *symmetric* if $\alpha = J_N \alpha J_N$. Equivalently, $\alpha \in S_N$ is symmetric if the graph of α is symmetric under rotation by π radians about the point $(\frac{N+1}{2}, \frac{N+1}{2})$.

Proposition 4.1. *Let $\alpha \in S_N$ and write α in simple form: $\alpha = \sigma[\beta_1, \beta_2, \dots, \beta_s]$. If α is symmetric then σ is necessarily symmetric and $\beta_{s+1-b} = J_{z_b}\beta_b J_{z_b}$ for all $b = 1, 2, \dots, s$. Consequently, if N is odd then $s = 2m + 1$ is odd, z_{m+1} is odd and $\beta_{m+1} \in S_{z_{m+1}}$ is symmetric. If N is even then then s may be even or odd; if, in addition, $s = 2m+1$ is odd then z_{m+1} is even and $\beta_{m+1} \in S_{z_{m+1}}$ is symmetric.*

Proof. The proof follows from the fact that, if $\alpha \in S_N$ is written in simple form as $\alpha = \sigma[\beta_1, \beta_2, \dots, \beta_s]$, then the simple form of $J_N \alpha J_N$ is

$$J_N \alpha J_N = (J_s \sigma J_s)[J_{z_s}\beta_s J_{z_s}, J_{z_{s-1}}\beta_{s-1}J_{z_{s-1}}, \dots, J_{z_1}\beta_1 J_{z_1}],$$

and the uniqueness of the simple form. □

Next we define an inflation operation which produces symmetric permutations. Let $0 \leqslant p \leqslant n$ and let $\{1, 2, \dots, n-p\} = U_1 \sqcup U_2 \sqcup \cdots \sqcup U_m$ be a decomposition into intervals. Put $z_b = |U_b|$ for $b = 1, 2, \dots, m$. Suppose that $\beta_b \in S_{z_b}$ for $b = 1, 2, \dots, m$ and $\beta_{m+1} \in S_{2p+1}$ or $\beta_{m+1} \in S_{2p}$. (For uniformity of notation we allow considering S_{2p} for $p = 0$; we will use $\varnothing$ to denote the "phantom" element of S_0.) Let $\sigma \in \begin{cases} S_{2m+1} & \text{if} \quad \beta_{m+1} \neq \varnothing \\ S_{2m} & \text{if} \quad \beta_{m+1} = \varnothing \end{cases}$,

We form the inflation

$$\alpha = \sigma[\beta_1, \beta_2, \dots, \beta_m, \beta_{m+1}, \beta_{m+2}, \dots, \beta_{2m+1}]$$

where $\beta_{2m+2-b} = J_{z_b}\beta_b J_{z_b}$ for $t = 1, 2, \dots, m$. Clearly $\alpha \in \begin{cases} S_{2n+1} & \text{if} \quad \beta_{m+1} \in S_{2p+1} \\ S_{2n} & \text{if} \quad \beta_{m+1} \in S_{2p} \end{cases}$ is a symmetric permutation. We call this operation *symmetric inflation* and denote it by

$$\alpha = \sigma[[\beta_1, \dots, \beta_m; \beta_{m+1}]].$$

Proposition 4.1 implies that the natural notion of a "simple symmetric permutation" is equivalent with the requirement that a symmetric permutation is simple. More precisely, we have the following corollary.

Corollary 4.2. *Let $\alpha \in S_{2n+1}$ (respectively, $\alpha \in S_{2n}$) be a symmetric element. Then α is simple in S_{2n+1} (respectively, in S_{2n}) if and only if*

$$\alpha = \sigma[[\beta_1, \beta_2, \ldots, \beta_m; \beta_{m+1}]]$$

implies $m = 0$ or $m = n$. □

Finally, Theorem 1.8 implies the existence of a *simple symmetric form expression* of a symmetric element $\alpha \in S_N$.

Proposition 4.3. *Let $\alpha \in S_N$ be symmetric. Then α can be written as*

$$\alpha = \sigma[[\beta_1, \ldots, \beta_m; \beta_{m+1}]],$$

where $\sigma \in S_M$ is simple with $M \geqslant 4$ or $\sigma = I_M$ or $\sigma = J_M$. Furthermore, this expression is unique if we require that M be maximal when $\sigma = I_M$ or $\sigma = J_M$. □

It also natural to discuss decomposing Δ_N^+ into symmetric inversion sets. Theorem 1.11 and Proposition 4.1 imply in a straightforward manner the following theorem.

Theorem 4.4. *Let $N = 2n + 1$ or $N = 2n$. Suppose $\alpha_1, \alpha_2, \ldots, \alpha_r \in S_N$ are symmetric elements and*

$$\Delta_N^+ = \Phi(\alpha_1) \sqcup \Phi(\alpha_2) \sqcup \cdots \sqcup \Phi(\alpha_r)$$

with all $\Phi(\alpha_a) \neq \emptyset$. Without loss of generality assume that the root $(1, N) \in \Phi(\alpha_1)$. Let $\alpha_1 = \sigma_1[[\beta_{11}, \beta_{12}, \ldots, \beta_{1m}; \beta_{1(m+1)}]]$ be the simple symmetric form expression for α_1 with $\sigma_1 \in S_M$ and a corresponding partition of the set $\{1, 2, \ldots, n\}$ into $m + 1$ intervals of lengths $z_1, z_2, \ldots, z_m, z_{m+1}$. Then, up to reordering of $\alpha_2, \alpha_3, \ldots, \alpha_r$ there exist unique elements $\sigma_a \in S_M$, $\beta_{ab} \in S_{z_b}$ and $\beta_{a(m+1)} \in S_P$, with $P = N - 2(z_1 + \ldots + z_m)$ such that $\alpha_a = \sigma_a[[\beta_{a1}, \beta_{a2}, \ldots, \beta_{am}; \beta_{a(m+1)}]]$ for $a = 2, \ldots, r$ and

(i)

$$\begin{aligned}
\Delta_M^+ &= \Phi(\sigma_1) \sqcup \Phi(\sigma_2) \sqcup \cdots \sqcup \Phi(\sigma_r), \\[1ex]
\Delta_{z_1}^+ &= \Phi(\beta_{11}) \sqcup \Phi(\beta_{21}) \sqcup \cdots \sqcup \Phi(\beta_{r1}), \\
\Delta_{z_2}^+ &= \Phi(\beta_{12}) \sqcup \Phi(\beta_{22}) \sqcup \cdots \sqcup \Phi(\beta_{r2}), \\
&\vdots \\
\Delta_{z_m}^+ &= \Phi(\beta_{1m}) \sqcup \Phi(\beta_{2m}) \sqcup \cdots \sqcup \Phi(\beta_{rm}), \\[1ex]
\Delta_P^+ &= \Phi(\beta_{1(m+1)}) \sqcup \Phi(\beta_{2(m+1)}) \sqcup \cdots \sqcup \Phi(\beta_{r(m+1)});
\end{aligned}$$

(ii) *if α_1 is minus-decomposable then $\sigma_1 = J$ and $\sigma_2 = \sigma_3 = \cdots = \sigma_r = I$;*

(iii) *if α_1 is minus-indecomposable then σ_1 is simple and, after relabelling $\alpha_2, \ldots, \alpha_r$, we have $\sigma_2 = J\sigma_1$, and $\sigma_3 = \sigma_4 = \cdots = \sigma_r = I$.*

In particular, σ_1 and at most one other of the σ_a are not equal to the identity.

Let q denote the number of σ_a which are not I, i.e., $q := \begin{cases} 1, & \textit{if } \alpha_1 \textit{ is minus-decomposable;} \\ 2, & \textit{if } \alpha_1 \textit{ is minus-indecomposable.} \end{cases}$
Again, after relabelling $\alpha_2, \ldots, \alpha_r$, we assume that $\sigma_{q+1} = \ldots = \sigma_r = I$.

The above decomposition of Δ_N^+ is irreducible if and only if the following four conditions hold

(i) *each of the decompositions listed in (i) above is irreducible;*
(ii) *exactly one of of $\beta_{a1}, \beta_{a2}, \ldots, \beta_{am}$ is not equal to the identity for $a = q+1, \ldots, r$;*
(iii) *$\beta_{ab} = I$ for $a = 1, \ldots, q$ and $b = 1, \ldots, m+1$;*
(iv) *$m = 1$ if α_1 is minus-decomposable.* □

5. Decompositions of Types B, C and D

We now turn to root systems of types B, C and D. We introduce some notation related to these root systems; our exposition is limited only to the minimum that we need. For a reference on root systems, see [H, Chap. III]. We will compare the root systems of types B, C and D with the root systems of type A. We take $\{e_1, e_2, \ldots, e_{n+1}\}$ as a standard basis for $\mathbb{R}^{n+1}$ and consider the Weyl group $\mathcal{W}(A_n) \cong S_{n+1}$ as the group of all permutations of this basis. With this notation, the positive roots are

$$\Delta^+_{A_n} = \{(i,j) = e_i - e_j \mid 1 \leqslant i < j \leqslant n+1\}.$$

To describe the root systems B_n, C_n and D_n, fix a standard basis $\{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n\}$ of $\mathbb{R}^n$. The corresponding positive roots are

$$\begin{array}{ll} B_n: & \Delta^+_{B_n} = \{\varepsilon_i \pm \varepsilon_j \mid 1 \leqslant i < j \leqslant n\} \sqcup \{\varepsilon_i \mid 1 \leqslant i \leqslant n\}; \\ C_n: & \Delta^+_{C_n} = \{\varepsilon_i \pm \varepsilon_j \mid 1 \leqslant i < j \leqslant n\} \sqcup \{2\varepsilon_i \mid 1 \leqslant i \leqslant n\}; \\ D_n: & \Delta^+_{D_n} = \{\varepsilon_i \pm \varepsilon_j \mid 1 \leqslant i < j \leqslant n\}. \end{array}$$

The Weyl group in each type is the group generated by reflections along roots. A reflection along $\varepsilon_i - \varepsilon_j$ is the same as transposing ε_i and ε_j, a reflection along $\varepsilon_i + \varepsilon_j$ is the same as transposing ε_i and ε_j while changing both signs, and a reflection along ε_i (or $2\epsilon_i$) is the same as changing the sign of ε_i.

The Lie algebras of types B_n, C_n, and D_n have classical descriptions as matrix algebras, and any $m \times m$ matrix has a natural action on an m-dimensional vector space. The weights of these respective natural representations are $\{\pm\varepsilon_i\}_{i=1}^n$ for C_n or D_n and $\{0\} \cup \{\pm\varepsilon_i\}_{i=1}^n$ for B_n. We thus obtain the following faithful permutation representations of the Weyl groups.

- The Weyl group $\mathcal{W}(B_n)$ is the set of sign-compatible permutations α of
$$\{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n, 0, -\varepsilon_n, \ldots, -\varepsilon_2, -\varepsilon_1\}.$$
That is, permutations obeying the condition that $\alpha(-\varepsilon_i) = -\alpha(\varepsilon_i)$ for all $1 \leqslant i \leqslant n$ and $\alpha(0) = 0$. Abstractly, $\mathcal{W}(B_n) \cong S_n \rtimes (\mathbb{Z}/2\mathbb{Z})^n$.

- The Weyl group $\mathcal{W}(C_n)$ is the set of sign-compatible permutations of the set
$$\{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n, -\varepsilon_n, \ldots, -\varepsilon_2, -\varepsilon_1\}.$$
Abstractly, $\mathcal{W}(C_n) \cong S_n \rtimes (\mathbb{Z}/2\mathbb{Z})^n$.
- The Weyl group $\mathcal{W}(D_n)$ is the set of sign-compatible permutations of the set
$$\{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n, -\varepsilon_n, \ldots, -\varepsilon_2, -\varepsilon_1\}$$
involving an even number of sign changes, i.e., permutations α for which $\alpha(\varepsilon_i) = -\varepsilon_j$ for an even number of indices $1 \leqslant i \leqslant n$. Abstractly, $\mathcal{W}(D_n) \cong S_n \rtimes (\mathbb{Z}/2\mathbb{Z})^{n-1}$.

Note that, even though the Weyl groups $\mathcal{W}(B_n)$ and $\mathcal{W}(C_n)$ are isomorphic, and their natural reflection actions on the vector space spanned by $\{\varepsilon_1, \ldots, \varepsilon_n\}$ are the same, we distinguish between them by their differing natural permutation representations.

In order to treat the roots systems of types B, C and D and their Weyl groups uniformly, we introduce some notation. Instead of discussing separately the root systems B_n, C_n or D_n we will sometimes discuss the root system X_n understanding that X stands for one of A, B, C or D. For uniformity of notation below, when considering X_n, we allow all values of $n \geqslant 0$: for instance, $\Delta^+_{X_0} = \emptyset$.

For $\alpha \in \mathcal{W}(X_n)$ we define $\Phi(\alpha) := \{v \in \Delta^+_{X_n} \mid \alpha(v) \notin \Delta^+_{X_n}\}$. It is clear that this definition of inversion set agrees with our previous definition when $X = A$. Let J_{X_n} denote the element of $\mathcal{W}(X_n)$ such that $\Phi(J_{X_n}) = \Delta^+_{X_n}$. As in the type A case, J_{X_n} is called the *longest element* of the corresponding Weyl group.

Most of the contents of §2 transfer to the cases when $X = B, C$ or D. For instance, call a set $\Phi \subset \Delta^+_{X_n}$ *closed* if $\alpha_1 + \alpha_2 \in \Phi$ whenever $\alpha_1, \alpha_2 \in \Phi$ and $\alpha_1 + \alpha_2 \in \Delta^+_{X_n}$. Proposition 2.1 still holds: $\Phi \subset \Delta^+_{X_n}$ is an inversion set if and only if both Φ and $\Delta^+_{X_n} \backslash \Phi$ are closed. Similarly, the obvious analogues of Lemmas 2.2 and 2.5, Proposition 2.7 and Corollary 2.8 hold in general.

We now turn to the basic problem of this paper, that of decomposing inversion sets, in the B_n, C_n, and D_n cases. Our idea is to relate it to the type A case by using the permutation representations of $\mathcal{W}(B_n)$, $\mathcal{W}(C_n)$ and $\mathcal{W}(D_n)$ listed above. Specifically, using the orders $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n, 0, -\varepsilon_n, \ldots, -\varepsilon_2, -\varepsilon_1$ for B_n, and $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n, -\varepsilon_n, \ldots, -\varepsilon_2, -\varepsilon_1$ for C_n and D_n, these permutation representations define embeddings of groups $\iota\colon \mathcal{W}(B_n) \hookrightarrow S_{2n+1}$, $\iota\colon \mathcal{W}(C_n) \hookrightarrow S_{2n}$ and $\iota\colon \mathcal{W}(D_n) \hookrightarrow S_{2n}$. (We use the same name for each of these homomorphisms, trusting that the name of the domain will make the homomorphism clear.)

In order to discuss the relationship between positive roots of X_n and S_{2n} (or S_{2n+1}), we set $\tilde{n} = 2n+1$ if $X_n = B_n$ and $\tilde{n} = 2n$ if $X_n = C_n$ or D_n, and set
$$\hat{\Delta}^+_{X_n} := \begin{cases} \Delta^+_{X_n} & \text{if} \quad X_n = C_n \\ \Delta^+_{X_n} \sqcup \{2\varepsilon_i \mid 1 \leqslant i \leqslant n\} & \text{if} \quad X_n = B_n \text{ or } D_n. \end{cases}$$

We define maps $\rho\colon \Delta^+_{\tilde{n}} \longrightarrow \hat{\Delta}^+_{X_n}$ by sending $e_1 \mapsto \varepsilon_1, \ldots, e_n \mapsto \varepsilon_n$, $e_{n+1} \mapsto -\varepsilon_n, \ldots, e_{2n} \mapsto -\varepsilon_1$ in cases C_n and D_n, and by sending $e_1 \mapsto \varepsilon_1, \ldots, e_n \mapsto \varepsilon_n$, $e_{n+1} \mapsto 0$, $e_{n+2} \mapsto -\varepsilon_n, \ldots, e_{2n+1} \mapsto -\varepsilon_n$ in case B_n.

Note that for $X = B$ or C we have $\iota(J_{X_n}) = J_{\tilde{n}}$, while for $X = D_n$ this is true if and only if n is even.

The following proposition establishes the behaviour of inversion sets under the maps ι and ρ above. Its proof is straightforward and is left to the reader.

Proposition 5.1. *Let $X = B, C$ or D.*

(i) *For any $\alpha \in \mathcal{W}(X_n)$, $\iota(\alpha) \in S_{\tilde{n}}$ is symmetric. If $X = B$ or C, the image of ι consists of all symmetric permutations in $S_{\tilde{n}}$; if $X = D$, the image of ι consists of all symmetric permutations $\beta \in S_{\tilde{n}}$ such that an even number of the elements $\beta(1), \dots, \beta(n)$ are greater than n.*

(ii) *The map ρ is surjective. More precisely, each element of $\hat{\Delta}^+_{X_n}$ of the form $2\varepsilon_i$ has a unique preimage in $\Delta^+_{\tilde{n}}$ and each of the remaining elements of $\hat{\Delta}^+_{X_n}$ has exactly two preimages in $\Delta^+_{\tilde{n}}$.*

(iii) *If $\beta \in S_{\tilde{n}}$ is symmetric then $\rho(\Phi(\beta)) \cap \Delta^+_{X_n} \subseteq \Delta^+_{X_n}$ is an inversion set.*

(iv) *If $\alpha \in \mathcal{W}(X_n)$ then $\Phi(\iota(\alpha)) = \rho^{-1}(\Phi(\alpha))$.*

(v) *Let $\alpha \in \mathcal{W}(X_n)$. If $X = B$ or C then $\beta = \iota(\alpha)$ is the unique element of $S_{\tilde{n}}$ such that $\Phi(\alpha) = \rho(\Phi(\beta)) \cap \Delta^+_{X_n}$. If $X = D$ there are exactly two elements $\beta = \iota(\alpha)$ and $\beta' \notin \iota(\mathcal{W}(X_n))$ such that $\Phi(\alpha) = \rho(\Phi(\beta)) \cap \Delta^+_{X_n} = \rho(\Phi(\beta')) \cap \Delta^+_{X_n}$.* □

Proposition 5.1 implies that the map ι interacts well with decompositions into inversion sets. More precisely, the following statements follow immediately from Proposition 5.1.

Corollary 5.2.

(i) *Assume that $\alpha_1, \alpha_2 \in \mathcal{W}(X_n)$ satisfy $\Phi(\iota(\alpha_1)) \cap \Phi(\iota(\alpha_2)) = \emptyset$. Then*

$$\rho(\Phi(\iota(\alpha_1)) \sqcup \Phi(\iota(\alpha_2))) = \rho(\Phi(\iota(\alpha_1))) \sqcup \rho(\Phi(\iota(\alpha_2))).$$

(ii) *Let $\alpha_1, \alpha_2, \dots, \alpha_r \in \mathcal{W}(X_n)$. Then*

$$\Delta^+_{X_n} = \bigsqcup_{i=1}^{r} \Phi(\alpha_r) \quad \text{if and only if} \quad \Delta^+_{\tilde{n}} = \bigsqcup_{i=1}^{r} \Phi(\iota(\alpha_r)).$$

(iii) *An element $\alpha \in \mathcal{W}(X_n)$ is irreducible if and only if $\iota(\alpha) \in S_{\tilde{n}}$ is irreducible.* □

Corollary 5.2 suggests that one can approach studying decompositions of $\Delta^+_{X_n}$ inversion sets by studying decompositions of $\Delta^+_{\tilde{n}}$ into symmetric inversion sets. Indeed, this approach can be carried out successfully in the cases when $X = B$ and C. Unfortunately, the ambiguity in Proposition 5.1 (i), (iv) prevented us from obtaining results for $X = D$. The first step is to define (or attempt to define) an inflation operation for the Weyl groups of types B, C and D. Proposition 5.1 (i) allows us to transfer the inflation operation for symmetric permutations to an inflation operation for the Weyl groups of types B and C but not D.

For the sake of completeness, below we provide the description of an inflation operations for the Weyl groups of types B and C. Let $X = B$ or C. Let $0 \leqslant p \leqslant n$ and let

$\{1, 2, \ldots, n-p\} = U_1 \sqcup U_2 \sqcup \cdots \sqcup U_m$ be a decomposition into intervals. Put $z_b = |U_b|$ for $b = 1, 2, \ldots, m$. Suppose that $\sigma \in \begin{cases} \mathcal{W}(B_m) & \text{if} \quad X_p \neq C_0 \\ \mathcal{W}(C_m) & \text{if} \quad X_p = C_0 \end{cases}$, $\beta_{m+1} \in \mathcal{W}(X_p)$ and $\beta_b \in S_{z_b}$ for $b = 1, 2, \ldots, m$. We form the inflation

$$\tilde{\alpha} = \iota(\sigma)[\beta_1, \beta_2, \ldots, \beta_m, \iota(\beta_{m+1}), \beta_{m+2}, \ldots, \beta_{2m+1}]$$

where $\beta_{2m+2-b} = J_{z_b}\beta_b J_{z_b}$ for $t = 1, 2, \ldots, m$. Note that in the case when $X_p = C_0$ the elements β_{m+1} and $\iota(\beta_{m+1})$ are actually empty and hence the expression above is well-defined. Then $\tilde{\alpha} \in S_{\tilde{n}}$ is symmetric and so $\tilde{\alpha} = \iota(\alpha)$ for a unique element $\alpha \in \mathcal{W}(X_n)$. We say that α is an inflation in $\mathcal{W}(X_n)$ and we write

$$\alpha = \sigma[[\beta_1, \beta_2, \ldots, \beta_m; \beta_{m+1}]]$$

to denote the fact that

$$\iota(\alpha) = \iota(\sigma)[\beta_1, \beta_2, \ldots, \beta_m, \iota(\beta_{m+1}), J_{z_m}\beta_m J_{z_m}, \ldots, J_{z_2}\beta_2 J_{z_2}, J_{z_1}\beta_1 J_{z_1}]$$

where $\alpha \in \mathcal{W}(X_n)$, $\sigma \in \begin{cases} \mathcal{W}(B_m) & \text{if} \quad X_p \neq C_0 \\ \mathcal{W}(C_m) & \text{if} \quad X_p = C_0 \end{cases}$, $\beta_b \in S_{z_b}$ for $b = 1, 2, \ldots, m$ and $\beta_{m+1} \in \mathcal{W}(X_p)$ with $z_1 + z_2 + \cdots + z_m = n - p$.

An element $\alpha \in \mathcal{W}(X_n)$ which cannot be realized as such an inflation in $\mathcal{W}(X_n)$ except with $m = 0$ or $m = n$ is said to be *simple in* $\mathcal{W}(X_n)$. Propositions 4.1 and 5.1(i) imply immediately the following statement.

Proposition 5.3. *Let $X = B$ or C and let $\alpha \in \mathcal{W}(X_n)$. Then α is simple in $\mathcal{W}(X_n)$ if and only if $\iota(\alpha)$ is simple in $S_{\tilde{n}}$.* □

We call the expression $\alpha = \sigma[[\beta_1, \beta_2, \ldots, \beta_m; \beta_{m+1}]]$ the *simple form expression for* $\alpha \in \mathcal{W}(X_n)$ if

$$\iota(\alpha) = \iota(\sigma)[\beta_1, \beta_2, \ldots, \beta_m, \iota(\beta_{m+1}), J_{z_m}\beta_m J_{z_m}, \ldots, J_{z_2}\beta_2 J_{z_2}, J_{z_1}\beta_1 J_{z_1}]$$

is the simple form expression for $\iota(\alpha)$ in $S_{\tilde{n}}$.

Note that the definition of inflation operation above does not apply for type D. On one hand, the element $\tilde{\alpha}$ defined above may not belong to the image of ι and, on the other hand, for $\alpha \in \mathcal{W}(D_n)$ the element $\iota(\alpha)$ may be an inflation

$$\tilde{\sigma}[\beta_1, \beta_2, \ldots, \beta_m, \tilde{\beta}_{m+1}, J_{z_m}\beta_m J_{z_m}, \ldots, J_{z_2}\beta_2 J_{z_2}, J_{z_1}\beta_1 J_{z_1}]$$

where $\tilde{\sigma}$ and $\tilde{\beta}_{m+1}$ are symmetric but not necessarily in the image of ι.

Theorem 5.4. *Let $X = B$ or C. Suppose $\alpha_1, \alpha_2, \ldots, \alpha_r \in \mathcal{W}(X_n)$ and*

$$\Delta^+_{X_n} = \Phi(\alpha_1) \sqcup \Phi(\alpha_2) \sqcup \cdots \sqcup \Phi(\alpha_r)$$

with all $\Phi(\alpha_a) \neq \emptyset$. Without loss of generality assume that the root $e_1 - e_{\tilde{n}} \in \Phi(\iota(\alpha_1))$. Let $\alpha_1 = \sigma_1[[\beta_{11}, \beta_{12}, \ldots, \beta_{1m}; \beta_{1(m+1)}]]$ be the simple form expression for α_1 with a corresponding partition of the set $\{1, 2, \ldots, n\}$ into $m+1$ intervals of lengths $z_1, z_2, \ldots, z_m, z_{m+1}$ where $z_{m+1} = p$. Then, up to reordering of $\alpha_2, \alpha_3, \ldots, \alpha_r$ there exist unique elements

$\sigma_a \in \begin{cases} \mathcal{W}(B_m) & \textit{if} \quad X_p \neq C_0 \\ \mathcal{W}(C_m) & \textit{if} \quad X_p = C_0 \end{cases}$, $\beta_{ab} \in S_{z_b}$ *and* $\beta_{a(m+1)} \in \mathcal{W}(X_p)$ *such that* $\alpha_a = \sigma_a[[\beta_{a1}, \beta_{a2}, \ldots, \beta_{am}; \beta_{a(m+1)}]]$ *for* $a = 2, \ldots, r$ *and*

(i)

$$\begin{aligned} \Delta^+_{X_m} &= \Phi(\sigma_1) \sqcup \Phi(\sigma_2) \sqcup \cdots \sqcup \Phi(\sigma_r), \\ \Delta^+_{A_{z_1-1}} &= \Phi(\beta_{11}) \sqcup \Phi(\beta_{21}) \sqcup \cdots \sqcup \Phi(\beta_{r1}), \\ \Delta^+_{A_{z_2-1}} &= \Phi(\beta_{12}) \sqcup \Phi(\beta_{22}) \sqcup \cdots \sqcup \Phi(\beta_{r2}), \\ &\vdots \\ \Delta^+_{A_{z_m-1}} &= \Phi(\beta_{1m}) \sqcup \Phi(\beta_{2m}) \sqcup \cdots \sqcup \Phi(\beta_{rm}), \\ \Delta^+_{X_p} &= \Phi(\beta_{1(m+1)}) \sqcup \Phi(\beta_{2(m+1)}) \sqcup \cdots \sqcup \Phi(\beta_{r(m+1)}); \end{aligned}$$

(ii) *if* α_1 *is minus-decomposable then* $\sigma_1 = J$ *and* $\sigma_2 = \sigma_3 = \cdots = \sigma_r = I$*;*

(iii) *if* α_1 *is minus-indecomposable then* σ_1 *is simple (in* $\mathcal{W}(B_m)$ *or in* $\mathcal{W}(C_m)$*) and, after relabelling* $\alpha_2, \ldots, \alpha_r$*, we have* $\sigma_2 = J\sigma_1$*, and* $\sigma_3 = \sigma_4 = \cdots = \sigma_r = I$*.*

In particular, σ_1 *and at most one other of the* σ_a *are not equal to the identity.*

Let q *denote the number of* σ_a *which are not* I*, i.e.,* $q := \begin{cases} 1, & \textit{if } \alpha_1 \textit{ is minus-decomposable}; \\ 2, & \textit{if } \alpha_1 \textit{ is minus-indecomposable}. \end{cases}$
Again, after relabelling $\alpha_2, \ldots, \alpha_r$*, we assume that* $\sigma_{q+1} = \ldots = \sigma_r = I$*.*

The above decomposition of $\Delta^+_{X_n}$ *is irreducible if and only if the following four conditions hold*

(i) *each of the decompositions listed in (i) above is irreducible;*

(ii) *exactly one of of* $\beta_{a1}, \beta_{a2}, \ldots, \beta_{am}$ *is not equal to the identity for* $a = q+1, \ldots, r$*;*

(iii) $\beta_{ab} = I$ *for* $a = 1, \ldots, q$ *and* $b = 1, \ldots, m+1$*;*

(iv) $m = 1$ *if* α_1 *is minus-decomposable.*

Proof. This result follows directly from Theorem 1.11 and the results of this section. Only two additional observations are needed. The first is that J_{X_m} is irreducible if and only if $m = 1$. The second is that the assumption $e_1 - e_{\tilde{n}} \in \Phi(\iota(\alpha_1))$ implies that $\iota(\sigma_1)$ is not the identity. $\square$

We conclude this section with a few remarks about decomposing $\Delta^+_{D_n}$ into inversion sets. As we already mentioned, it is not clear how to define the inflation operation for type D. Another possible approach to decomposing $\Delta^+_{D_n}$ may be to use the fact that $\Delta^+_{D_n}$ embeds naturally into $\Delta^+_{C_n}$. Indeed, one can show that every decomposition of $\Delta^+_{C_n}$ into inversion sets produces a unique decomposition of $\Delta^+_{D_n}$ into inversion sets. We do not know, however, whether the converse is true.

6. Enumerative Results

The inductive description for a decomposition provided by Theorems 1.11 and 5.4 allows us to use generating series or recursion to enumerate many different types of decompositions. We give a few examples.

Let s_n be the number of simple pairs in S_n, i.e., the number of sets $\{\alpha, J\alpha\}$ with $\alpha \in S_n$ and both α and $J\alpha$ simple (note that by Lemma 2.4 α is simple if and only if $J\alpha$ is simple). Let $S_A(z) = \sum_{n\geqslant 0} s_n z^n = z^2 + z^4 + 3z^5 + \cdots$ be the corresponding generating function. By [AAK, page 5] we have the following description of $S(z)$. Let $F(z) = \sum_{n\geqslant 1} n! z^n$ and $G(z) = \sum_{n\geqslant 1} g_n z^n$ its functional inverse, i.e., the function defined by the relation $G(F(z)) = z$. Then $s_1 = 0$, $s_2 = 1$, and $s_n = -g_n/2 - (-1)^n$ for $n \geqslant 3$.

Number of decompositions into irreducibles. Let a_n be the number of decompositions $\Delta_n^+ = \Phi(\alpha_1) \sqcup \Phi(\alpha_2) \sqcup \cdots \sqcup \Phi(\alpha_r)$ into non-empty inversion sets, where each $\alpha_k \in S_n$ is irreducible, and where we ignore the order in the decomposition. Let $A(z) = \sum_{n\geqslant 1} a_n z^n$ be the generating series. Theorem 1.11 leads to the relation $A(z) = S_A(A(z)) + z$, which recursively determines the coefficients a_n. Here are the low order terms of $A(z)$:

$$A(z) = z + z^2 + 2z^3 + 6z^4 + 23z^5 + 114z^6 + 717z^7 + 5510z^8 + 49570z^9 + 504706z^{10} + \cdots .$$

Unfortunately we were not able to obtain a closed form expression for the coefficients a_n.
Decompositions of maximal length. If $\alpha \neq I$ then the inversion set $\Phi(\alpha)$ must contain at least one simple root. Since there are only $n-1$ simple roots, any decomposition $\Delta_n^+ = \Phi(\alpha_1) \sqcup \Phi(\alpha_2) \sqcup \cdots \sqcup \Phi(\alpha_r)$, with no $\alpha_s = I$ must satisfy $r \leqslant n-1$. Let $\mathrm{Cat}_\mathrm{A}(n-1)$ denote the number of decompositions of Δ_n^+ into exactly $n-1$ non-empty inversion sets. (Thus each inversion set appearing in the decomposition must contain exactly one simple root.)

Lemma 6.1. *$Cat_A(n) = \frac{1}{n+1}\binom{2n}{n}$, the n^{th} Catalan number.*

Proof. We consider decompositions of the form $\Delta_n^+ = \Phi(\alpha_1) \sqcup \Phi(\alpha_2) \sqcup \cdots \sqcup \Phi(\alpha_{n-1})$ and compute $\mathrm{Cat}_\mathrm{A}(n-1)$. Without loss of generality, the highest root $e_1 - e_n \in \Phi(\alpha_1)$. Suppose that $e_k - e_{k+1}$ is the simple root in $\Phi(\alpha_1)$. Then $\alpha_1(k+1) < \alpha_1(k+2) < \cdots < \alpha_1(n) < \alpha_1(1) < \alpha_1(2) \cdots < \alpha_1(k)$ and therefore $\alpha_1 = (n-k+1, n-k+2, \ldots, n, 1, 2 \ldots, n-k) = (1,2)[I_k, I_{n-k}]$. Let $U_1 := \{1, 2, \ldots, k\}$ and $U_2 := \{k+1, k+2, \ldots, n\}$. Then $\Phi(\alpha_1) = \{(e_i - e_j \in \Delta_n^+ \mid i \in U_1, j \in U_2\} = \{e_i - e_j \in \Delta_n^+ \mid i \leqslant k, j \geqslant k+1\}$. Therefore $\Delta_{U_1}^+ \sqcup \Delta_{U_2}^+ = \Phi(\alpha_2) \sqcup \Phi(\alpha_3) \sqcup \cdots \sqcup \Phi(\alpha_{n-1})$. Without loss of generality, $\Delta_{U_1}^+ = \Phi(\alpha_2) \sqcup \Phi(\alpha_3) \sqcup \cdots \sqcup \Phi(\alpha_{k-1})$ and $\Delta_{U_2}^+ = \Phi(\alpha_{k+1}) \sqcup \Phi(\alpha_{k+2}) \sqcup \cdots \sqcup \Phi(\alpha_{n-1})$. This yields the recursion relation $\mathrm{Cat}_\mathrm{A}(n-1) = \sum_{t=1}^{n-1} \mathrm{Cat}_\mathrm{A}(t-1)\mathrm{Cat}_\mathrm{A}(n-t-1) = \sum_{t=0}^{n-2} \mathrm{Cat}_\mathrm{A}(t)\mathrm{Cat}_\mathrm{A}(n-t-2)$. Thus $\mathrm{Cat}_\mathrm{A}(n) = \sum_{t=1}^{n} \mathrm{Cat}_\mathrm{A}(t-1)\mathrm{Cat}_\mathrm{A}(n-t)$. Since $\mathrm{Cat}_\mathrm{A}(1) = 1$ and $\mathrm{Cat}_\mathrm{A}(2) = 2$ we see that $\mathrm{Cat}_\mathrm{A}(n)$ satisfies the usual recursion relation for the Catalan numbers. $\square$

This incarnation of the Catalan numbers is number 186 in Richard Stanley's monograph [S].

Type B/C results. Theorem 5.4 leads to similar recursions in types B/C. Let $S_B(z)$ be the generating series for the number of simple pairs in type B_n/C_n. Equivalently the coefficient of z^n in $S_B(z)$ is the number of pairs of simple elements in S_{2n+1} each of which are symmetric. The isomorphism $\mathcal{W}(B_n) \cong \mathcal{W}(C_n)$ implies that this is also the number of pairs of simple symmetric elements in S_{2n}. One deduces the functional equation

$$S_B(F(z)) = 1 - \frac{1}{1+F(2z)} - \frac{2F(z)}{1+F(z)},$$

(where $F(z) = \sum_{n\geqslant 1} n! z^n$ as above) which determines $S_B(z)$. Here are some low order terms:

$$S_B(z) = 2z^2 + 10z^3 + 90z^4 + 966z^5 + 12338z^6 + 181470z^7 + 3018082z^8 + 55995486z^9 + \cdots .$$

Decompositions into irreducibles. Let b_n be the number of decompositions of the positive roots in types B_n/C_n into disjoint unions of irreducible inversion sets, and let $B(z) = \sum_{n\geqslant 1} b_n z^n$ to be the generating function. Theorem 5.4 leads to the relation

$$B(z) = \frac{S_B(A(z))}{1 - S_B(A(z))},$$

which completely determines $B(z)$. Here are the low order terms of $B(z)$:

$$B(z) = z + 3z^2 + 14z^3 + 100z^4 + 973z^5 + 11804z^6 + 168809z^7 + 2757930z^8 + 50522912z^9 + \cdots .$$

$\mathbf{B}_n/\mathbf{C}_n$ Catalan numbers. Let $\mathrm{Cat}_{\mathrm{B}}(n)$ be the number of decompositions of the positive roots of B_n/C_n into disjoint unions of inversion sets, where each inversion set contains a single simple root. The isomorphism $\mathcal{W}(B_n) \cong \mathcal{W}(C_n)$ implies that the number of such decompositions is the same for types B_n and C_n. As in type A, these are the decompositions of maximal length (subject to the restriction that each inversion set is non-empty) and thus are irreducible decompositions.

Proposition 6.2. *The numbers $Cat_B(n)$ satisfy the recursion $Cat_B(n) = Cat_B(n-1) + 2\sum_{k=0}^{n-2} Cat_A(n-k-1)Cat_B(k)$, and thus*

$$\sum_{n\geqslant 1} Cat_B(n) z^n = \frac{1}{(1-4z)^{\frac{1}{2}} + z}.$$

Proof. We consider the B_n case. Suppose then that $\Delta^+_{B_n} = \Phi(\alpha_1) \sqcup \Phi(\alpha_2) \sqcup \cdots \sqcup \Phi(\alpha_n)$ where each $\alpha_i \in \mathcal{W}(B_n)$ and each $\Phi(\alpha_i)$ contains a single simple root of $\Delta^+_{B_n}$. Without loss of generality $\Phi(\iota(\alpha_1))$ contains $e_1 - e_{2n+1}$. By Theorem 5.4, we have $\alpha_1 = \sigma_1[[\beta_{11}, \beta_{12}, \ldots, \beta_{1s}; \beta_{1(s+1)}]]$ where $\sigma_1 \in \mathcal{W}(B_s)$ and either $\sigma_1 = J_{B_1}$ or σ_1 is simple and $\Phi(\sigma_1)$ contains a single simple root. Thus if $\sigma_1 \neq J_{B_1}$ then $\iota(\sigma_1)$ is simple, symmetric and $\Phi(\iota(\sigma_1))$ contains a pair of A_{2n} simple roots of the form $e_i - e_{i+1}, e_{i'-1} - e_{i'}$, where $i' = 2n+2-i$. It is not hard to see that this forces $\iota(\sigma_1) = J_3$, $\iota(\sigma_1) = (41352)$ or

$\iota(\sigma_1) = (25314)$. The last possibility is excluded by the fact that $\Phi(\iota(\sigma_1))$ contains the highest root $e_1 - e_5$.

First suppose that $\iota(\sigma_1) = J_3$ and let $\{1, 2, \dots, 2n+1\} = U_1 \sqcup U_2 \sqcup U_3$ be the corresponding decomposition into intervals with $|U_1| = |U_3| = n - k$ and $|U_2| = 2k + 1$ where $0 \leqslant k \leqslant n - 1$. Then $\iota(\alpha_j) = I_3[\beta_{j1}, \beta_{j2}, \beta_{j3}]$ for $j = 2, 3, \dots, n$. Furthermore, without loss of generality, $\Delta^+_{U_1} = \Phi(\beta_{21}) \sqcup \Phi(\beta_{31}) \sqcup \cdots \sqcup \Phi(\beta_{(n-k)1})$ is a maximal length decomposition of a root system of type A_{n-k-1}. There are $\mathrm{Cat}_{\mathrm{A}}(n-k-1)$ such decompositions. (We also have $\Delta^+_{U_3} = \Phi(J\beta_{23}J) \sqcup \Phi(J\beta_{33}J) \sqcup \cdots \sqcup \Phi(J\beta_{(n-k)3}J)$.) Finally $\Delta^+_{U_2} = \Phi(\beta_{(n-k+1)2}) \sqcup \Phi(\beta_{(n-k+2)2}) \sqcup \cdots \sqcup \Phi(\beta_{n2})$ is a maximal symmetric decomposition. There are $\mathrm{Cat}_{\mathrm{B}}(k)$ such decompositions. Thus there are $\sum_{k=0}^{n-1} \mathrm{Cat}_{\mathrm{A}}(n-k-1)\mathrm{Cat}_{\mathrm{B}}(k)$ maximal decompositions of $\Delta^+_{B_n}$ with $\iota(\alpha_1) = J_3$.

Next suppose that $\iota(\sigma_1) = (41352)$ and let $\{1, 2, \dots, 2n+1\} = U_1 \sqcup U_2 \sqcup \cdots \sqcup U_5$ be the corresponding decomposition. Then, as above, $\iota(\alpha_2) \sqcup \iota(\alpha_3) \sqcup \cdots \sqcup \iota(\alpha_n)$ comprises maximal A type decompositions of $\Delta^+_{U_1}$ and $\Delta^+_{U_2}$ and a maximal symmetric decomposition of $\Delta^+_{U_3}$. Thus there are

$$\begin{aligned}
\sum_{z_1=1}^{n-1} \sum_{z_2=1}^{n-z_1} \mathrm{Cat}_{\mathrm{A}}(z_1)\mathrm{Cat}_{\mathrm{A}}(z_2)\mathrm{Cat}_{\mathrm{B}}(n - z_1 - z_2) &= \sum_{k=0}^{n-2} \sum_{z_1+z_2=n-k} \mathrm{Cat}_{\mathrm{B}}(k)\mathrm{Cat}_{\mathrm{A}}(z_1)\mathrm{Cat}_{\mathrm{A}}(z_2) \\
&= \sum_{k=0}^{n-2} \mathrm{Cat}_{\mathrm{B}}(k)\mathrm{Cat}_{\mathrm{A}}(n-k-1)
\end{aligned}$$

maximal decompositions of $\Delta^+_{B_n}$ with $\iota(\alpha_1) = (41352)$.

Adding the contributions of the two cases gives

$$\mathrm{Cat}_{\mathrm{B}}(n) = \mathrm{Cat}_{\mathrm{B}}(n-1) + 2\sum_{k=0}^{n-2} \mathrm{Cat}_{\mathrm{A}}(n-k-1)\mathrm{Cat}_{\mathrm{B}}(k)$$

as claimed. This easily implies the stated form of the generating function. □

These numbers appear in other combinatorial settings, see A081696 in the Sloane on-line encyclopedia of integer sequences.

Remark. We have chosen to call these numbers the "type B/C Catalan numbers", since they come from an enumerative problem about Coxeter groups which yields the usual Catalan numbers in the type A case. There is at least one other use of the term "Catalan numbers for other types" in the literature, again stemming from an enumerative problem (generalizing non-crossing partitions) valid for all Coxeter groups. In this second problem, the type B_n/C_n numbers are $\binom{2n}{n}$ (see [Arm, pg. 39]) – different from the numbers given by the recursion and generating function above.

Number of decompositions into triples. The most important case – in any type – of the problems motivating these questions about decompositions is the case of decompositions into a disjoint union of three inversion sets. As described in §1.2 this corresponds to the the case of the eigenvalues of three Hermitian matrices summing to zero (respectively the cup product of two cohomology groups into a third, after a similar symmetrization). The corresponding enumerative/classification problem is to write down all triples α_1, α_2, $\alpha_3 \in S_n$ (again disregarding order) with $\Delta_n^+ = \Phi(\alpha_1) \sqcup \Phi(\alpha_2) \sqcup \Phi(\alpha_3)$. We make the further restriction that no $\alpha_j = I$ (all such triples are of the form (w, Jw, I) and hence elementary to understand). Theorems 1.11 and 5.4 provide a recursive way to generate and enumerate all such triples. Briefly, the method is a parallel recursion keeping track of not only the triples of the kind above, but also the subset of those triples where $\alpha_1 = J_m$ for some m. At each step, the new triples of each kind depend on the triples of both kinds for smaller n. (We omit the exact description of the recursion since, although elementary, it is slightly messy.) Here is a small table of the number of such triples, and both the A_n and B_n/C_n cases.

n	A_n triples	B_n/C_n triples
1		1
2	1	4
3	3	33
4	17	351
5	129	4210
6	1116	55495
7	10474	800476
8	104604	12654164
9	1101012	219870187
10	12153179	4206375350
11	140397525	88539459103
12	1697555983	2043502238365
13	21516940295	51440876843396
14	286680892462	1403608329020473
15	4028129552836	41257592671098146
16	59885247963954	1299045890821350162
17	944511887685826	43596718839825553381
18	15828354015222453	1552871403021630700936
19	281880601827533671	58488502832975791077421
20	5327985147037232973	2322044948865982864468235

7. Decomposing a single inversion set

In this section we provide a recursive algorithm for listing all decompositions of the inversion set $\Phi(\alpha)$ of a given element $\alpha \in S_n$ as

$$\Phi(\alpha) = \Phi(\alpha_1) \sqcup \Phi(\alpha_2)$$

and provide a formula for the number of such decompositions[3].

Let $\alpha = \sigma[\beta_1, \dots, \beta_m]$ be the simple form of α. To list all ordered decompositions[4] of $\Phi(\alpha)$ we proceed as follows:

Step 1. Write all decompositions

$$\Phi(\beta_b) = \Phi(\beta_{1b}) \sqcup \Phi(\beta_{2b}) \qquad \text{for} \qquad 1 \leqslant b \leqslant m.$$

Step 2. For every decomposition of Step 1 write the decompositions

$$\begin{array}{rcl} \alpha_1 & = & \sigma[\beta_{11}, \dots, \beta_{1m}] \\ \alpha_2 & = & I_m[\beta_{21}, \dots, \beta_{2m}] \end{array}$$

and, if $\sigma \neq I_m$,

$$\begin{array}{rcl} \alpha_1 & = & I_m[\beta_{11}, \dots, \beta_{1m}] \\ \alpha_2 & = & \sigma[\beta_{21}, \dots, \beta_{2m}]. \end{array}$$

If $\sigma = I_m$ or if σ is simple with $m \geqslant 4$, these are all decompositions of $\Phi(\alpha)$ and the algorithm stops. The remaining possibility is $\sigma = J_m$, and in this case we continue to the next step.

Step 3. Write all partitions $\mathcal{U}$ of the set $\{1, 2, \dots, m\}$ into $l \geqslant 4$ intervals $U_1, U_2, \dots, U_l$ of lengths $z_1, z_2, \dots, z_l$ and for each such partition construct the following elements:

$$\begin{array}{rcl} \gamma_1 & = & J_{z_1}[\beta_1, \dots, \beta_{z_1}]; \\ \gamma_2 & = & J_{z_2}[\beta_{z_1+1}, \dots, \beta_{z_1+z_2}]; \\ & \vdots & \\ \gamma_l & = & J_{z_l}[\beta_{z_1+\dots+z_{l-1}+1}, \dots, \beta_m]. \end{array}$$

Step 4. Write all decompositions

$$\Phi(\gamma_c) = \Phi(\gamma_{1c}) \sqcup \Phi(\gamma_{2c}) \qquad \text{for} \qquad 1 \leqslant c \leqslant l.$$

Step 5. For every decomposition of Step 1 and every simple $\sigma \in S_l$ write the decompositions

$$\begin{array}{rcl} \alpha_1 & = & \sigma[\gamma_{11}, \dots, \gamma_{1l}] \\ \alpha_2 & = & (J_l\sigma)[\gamma_{21}, \dots, \gamma_{2l}] \end{array} .$$

These complete the list of all decompositions of $\Phi(\alpha)$.

The algorithm above provides a recursive formula for the number of ordered decompositions of $\Phi(\alpha)$. For $\alpha \in S_n$, denote by $d_2(\alpha)$ the number of ordered decompositions of $\Phi(\alpha)$

[3] We thank Lukas Katthän for asking us this question after a previous version of this paper appeared on ArXiv, see [LK].

[4] We choose to list the ordered decompositions of $\Phi(\alpha)$ to simplify the formula for counting them.

into two pieces as above. With this notation $d_2(I) = 1$ and $d_2(\sigma) = 2$ if σ is simple. As in §6 let s_l denote the number of simple pairs in S_l, so that $2s_l$ is the number of simple elements. Then, in the notation of the algorithm, one has the following formula for $d_2(\alpha)$:

$$d_2(\alpha) = \begin{cases} d_2(\beta_1)\cdots d_2(\beta_m) & \text{if } \sigma = I_m \\ 2d_2(\beta_1)\cdots d_2(\beta_m) & \text{if } \sigma \text{ is simple and } m \geqslant 4 \\ 2d_2(\beta_1)\cdots d_2(\beta_m) + 2\sum_{l\geqslant 4} s_l(\sum_{\mathcal{U}} d_2(\gamma_1)\ldots d_2(\gamma_l)) & \text{if } \sigma = J_m, \end{cases}$$

where the summation $\sum_{\mathcal{U}}$ in the third case is over all partitions $\mathcal{U}$ of $\{1, 2, \ldots, m\}$ into l intervals.

The problem of decomposing a single inversion set can be solved algorithmically for types B and C as well and, furthermore, one can also discuss the decomposition of a given inversion set into the disjoint union of a fixed number of inversion sets. These descriptions are analogous to the one given above and we omit them here.

8. Parametrizing regular codimension n faces of the Littlewood-Richardson cone

The Horn conjecture (now a theorem due to Klyachko and Knutson and Tao) describes the Littlewood-Richardson cone in terms of its supporting hyperplanes. The article [F] is a good reference for the history and solution of this conjecture. One may also seek to describe the cone in terms of its generating rays. In this section we explain how our work allows us to find the extremal rays lying on the smallest regular faces of the cone. In addition, this method proves that these faces are simplicial. For clarity of exposition we discuss only the case of type A but everything carries over to the cases of types B and C.

Regular faces of the Littlewood-Richardson cone. To describe how our results relate to the Littlewood-Richardson cone we first convert the problem of eigenvalues of Hermitian matrices to its symmetric version as in §1.2. I.e., instead of Hermitian matrices A, B, C satisfying $C = A+B$ we will consider Hermitian matrices A, B, C satisfying $A+B+C = 0$. It is clear that the cone $\mathcal{C}''$ of such triples is contained in the hyperplane V defined by the trace condition

$$\lambda_1 + \ldots + \lambda_n + \mu_1 + \ldots + \mu_n + \nu_1 + \ldots + \nu_n = 0,$$

and contains the two-dimensional subspace $W \subset V$ of $(\mathbb{R}^n)^3$ spanned by $(1, \ldots, 1, 0, \ldots, 0, -1, \ldots, -1)$ and $(0, \ldots, 0, 1, \ldots, 1, -1, \ldots, -1)$. Denote by $\mathcal{C}$ the image of $\mathcal{C}''$ under the projection $V \to V/W$. We will use again (λ, μ, ν) to denote the projection of a point in V to V/W. The natural coordinates in V/W are $\lambda = (a_1, \ldots, a_{n-1})$, $\mu = (b_1, \ldots, b_{n-1})$, and $\nu = (c_1, \ldots, c_{n-1})$, where $a_i = \lambda_i - \lambda_{i+1}$, $b_i = \mu_i - \mu_{i+1}$, and $c_i = \nu_i - \nu_{i+1}$ for $1 \leqslant i \leqslant n-1$. Clearly $V/W \cong (\mathbb{R}^{n-1})^3$ and S_n acts naturally on each of the components of $(\mathbb{R}^{n-1})^3$: we fix the natural basis $\{e_i - e_{i+1} \,|\, 1 \leqslant i \leqslant n-1\}$ of $\mathbb{R}^{n-1}$ and the action of S_n is by permuting the indices of this basis. The cone $\mathcal{C}$ is a pointed polyhedral cone of full dimension. Each of the coordinate hyperplanes $a_i = 0$, $b_i = 0$, and $c_i = 0$ for a fixed i with $1 \leqslant i \leqslant n-1$ is a facet of $\mathcal{C}$. Let $\mathbb{R}^{3n-3}_{\geqslant 0} \subset (\mathbb{R}^{n-1})^3$ denote the dominant cone defined by $a_i \geqslant 0$, $b_i \geqslant 0$, $c_i \geqslant 0$ for all $1 \leqslant i \leqslant n-1$. A face of $\mathcal{C}$ is called *regular* if it intersects the interior of $\mathbb{R}^{3n-3}_{\geqslant 0}$. N. Ressayre proved that the regular faces of $\mathcal{C}$ have codimension at most $n-1$. Furthermore, the faces of codimension $n-1$ are exactly the intersection of $\mathbb{R}^{3n-3}_{\geqslant 0}$ with the codimension $n-1$ subspaces $T_{\alpha_1,\alpha_2,\alpha_3}$ defined by

$$\alpha_1^{-1}\lambda + \alpha_2^{-1}\mu + \alpha_3^{-1}\nu = 0$$

for $(\alpha_1, \alpha_2, \alpha_3)$ with the property that $\Delta_n^+ = \Phi(\alpha_1) \sqcup \Phi(\alpha_2) \sqcup \Phi(\alpha_3)$, see [R, Theorem C]. Let $(\alpha_1, \alpha_2, \alpha_3)$ be such a triple and denote by $\mathcal{C}_{\alpha_1,\alpha_2,\alpha_3}$ the corresponding face of $\mathcal{C}$, i.e., $\mathcal{C}_{\alpha_1,\alpha_2,\alpha_3} = T_{\alpha_1,\alpha_2,\alpha_3} \cap (\mathbb{R}^{n-1})^3_+ = T_{\alpha_1,\alpha_2,\alpha_3} \cap \mathcal{C}$.

Note that $\mathcal{C}_{\alpha_1,\alpha_2,\alpha_3}$ is described by its $(n-1)$ defining hyperplanes, given by the equation $\alpha_1^{-1}\lambda + \alpha_2^{-1}\mu + \alpha_3^{-1}\nu = 0$ and the conditions that it lies in the dominant chamber, namely that $a_i \geqslant 0$, $b_i \geqslant 0$, and $c_i \geqslant 0$. It is difficult to conclude from this description what its defining rays are. We will now show that Theorem 1.11 allows us to conclude that $\mathcal{C}_{\alpha_1,\alpha_2,\alpha_3}$ is a simplicial cone and provides an algorithm for writing down its defining rays. (The fact that $\mathcal{C}_{\alpha_1,\alpha_2,\alpha_3}$ is a simplicial cone also follows from some results in [DR].) In this section it will be convenient to identify the elements of Δ_n^+ with the vectors $e_i - e_j$. Consider the inner product in $\mathbb{R}^{n-1}$ defined by $(\lambda, e_i - e_j) := \lambda_i - \lambda_j$. It is immediate that, for $1 \leqslant i < j \leqslant n$,

$$(\lambda, e_i - e_j) = a_i + \ldots + a_{j-1}.$$

This inner product is S_n-invariant; in particular we have $(\alpha^{-1}\lambda, e_i - e_j) = (\lambda, \alpha(e_i - e_j))$ for any $\alpha \in S_n$ and $e_i - e_j \in \Delta_n^+$. To obtain a set of defining equations for $T_{\alpha_1,\alpha_2,\alpha_3}$ it is sufficient to chose a basis $\{v_1, v_2, \ldots, v_{n-1}\}$ of $\mathbb{R}^{n-1}$ consisting of elements of Δ_n^+ and write

$$(\alpha_1^{-1}\lambda + \alpha_2^{-1}\mu + \alpha_3^{-1}\nu, v_i) = 0$$

for $1 \leqslant i \leqslant n-1$. Consider the form of the equation $(\alpha_1^{-1}\lambda + \alpha_2^{-1}\mu + \alpha_3^{-1}\nu, v) = 0$ for $v \in \Delta_n^+$. Exactly one of the roots $\alpha_1(v), \alpha_2(v), \alpha_3(v)$ is negative, say $\alpha_1(v) = -(e_i - e_j)$, $\alpha_2(v) = e_k - e_l$, and $\alpha_3(v) = e_p - e_q$. Then $(\alpha_1^{-1}\lambda + \alpha_2^{-1}\mu + \alpha_3^{-1}\nu, v) = 0$ becomes

$$a_i + \cdots + a_{j-1} = b_k + \cdots + b_{l-1} + c_p + \ldots + c_{q-1}.$$

This equation is especially simple when $-w_1(v)$ is a simple root, i.e., when $j = i+1$. Then it becomes

$$a_i = b_k + \cdots + b_{l-1} + c_p + \cdots + c_{q-1}.$$

Borrowing from elementary linear algebra, we call a_i the *v-pivot variable* and $b_k, \ldots, b_{l-1}$, $c_p, \ldots, c_{q-1}$ *v-free variables* in this case.

Proposition 8.1. *Assume that $\Delta_n^+ = \Phi(\alpha_1) \sqcup \Phi(\alpha_2) \sqcup \Phi(\alpha_3)$. The set*

$$S_{\alpha_1,\alpha_2,\alpha_3} = \{v \in \Delta_n^+ \mid -\alpha_i(v) \text{ is a simple root for some } 1 \leqslant i \leqslant 3\}.$$

is a basis of $\mathbb{R}^{n-1}$. Furthermore, this set can be labeled $\{v_1, v_2, \ldots, v_{n-1}\}$ so that, for $i < j$, the v_i-pivot variable is not a v_j-free variable.

Proof. Let $\alpha_i = \sigma_i[\beta_{i1}, \beta_{i2}, \dots, \beta_{im}]$, $i = 1, 2, 3$, be the expression of the α_i in terms of inflations guaranteed by Theorem 1.11, and let $\{1, 2, \dots, n\} = U_1 \sqcup U_2 \sqcup \dots \sqcup U_m$ be the corresponding decomposition into intervals. Assume $v = e_i - e_j \in S_{\alpha_1,\alpha_2,\alpha_3}$. Define the *level of* v inductively as follows: if i and j belong to different parts of $\{1, \dots, n\}$, then the level of v is one; otherwise, $i, j \in U_k$ and the level of v is one plus the level of v for the decomposition $\Delta^+_{z_k} = \Phi(\beta_{1k}) \sqcup \Phi(\beta_{2k}) \sqcup \Phi(\beta_{3k})$. Consider the projection $\{1, 2, \dots, n\} \to \{1, 2, \dots, m\}$ induced by the decomposition into intervals. Under this projection the level one elements of $S_{\alpha_1,\alpha_2,\alpha_3}$ are sent bijectively to the elements of $S_{\sigma_1,\sigma_2,\sigma_3}$, and these latter elements form a basis of $\mathbb{R}^{m-1}$ since, by Theorem 1.11, $(\sigma_1, \sigma_2, \sigma_3) = (\sigma_1, J\sigma_1, I)$, and for such a triple the statement about being a basis is easily checked. The elements of level greater than one are sent to zero under this map. On the other hand, by induction, the elements of level greater than one form a basis of the subspace generated by $\{e_i - e_j \mid i, j \text{ in the same } U_k\}$. Combining the above we conclude that $S_{\alpha_1,\alpha_2,\alpha_3}$ is a basis of $\mathbb{R}^{n-1}$.

To prove the second assertion, we order $S_{\alpha_1,\alpha_2,\alpha_3}$ linearly so that elements of lower level come before elements of higher level. Notice first that if v_1 is of level one and v_2 is of level greater than one, than no v_1-pivot variable is v_2-free. Now assume that both v_1 and v_2 are of level one. Passing to the projection as above, we conclude again that no v_1-pivot variable is v_2-free. $\square$

We call the v_i-pivot variables simply *pivot variables* of $C_{\alpha_1,\alpha_2,\alpha_3}$ and the rest of a_i, b_i, c_i we call *free variables.*

Corollary 8.2. *$C_{\alpha_1,\alpha_2,\alpha_3}$ is a simplicial cone.*

Proof. It follows from Proposition 8.1 that there are exactly $n - 1$ pivot variables. Furthermore, by ordering them as above we can start from the bottom and replace any pivot variable appearing in the expression of another pivot variable by its expression. When we reach the top equation, every pivot variable will have become expressed with non-negative coefficients in terms of the free variables only. $\square$

Example 8.3. We continue with Example 1.12. Recall that $\alpha_1 = (4, 5, 6, 1, 7, 8, 3, 2)$, $\alpha_2 = (5, 3, 4, 8, 1, 2, 6, 7)$, $\alpha_3 = (1, 3, 2, 4, 6, 5, 7, 8)$ and $\Delta^+_8 = \Phi(\alpha_1) \sqcup \Phi(\alpha_2) \sqcup \Phi(\alpha_3)$. The set $S_{\alpha_1,\alpha_2,\alpha_3}$ together with the corresponding equations by level is:

$$
\begin{array}{llll}
\text{Level 1:} & e_2 - e_6 : & a_2 = b_5 + b_6 + b_7 + & c_3 + c_4 \\
 & e_4 - e_8 : & a_7 = b_1 + & c_4 + c_5 + c_6 + c_7 \\
 & e_1 - e_7 : & b_3 = a_5 + & c_1 + c_2 + c_3 + c_4 + c_5 + c_6 \\
\text{Level 2:} & e_1 - e_3 : & a_4 = b_4 + b_5 + & c_1 \\
 & e_5 - e_6 : & c_5 = a_1 + & b_7 \\
 & e_7 - e_8 : & b_2 = a_6 + & c_7 \\
\text{Level 3:} & e_2 - e_3 : & c_2 = a_3 + & b_5.
\end{array}
$$

The pivot variables c_2 and c_5 appear in the expressions for a_7 and b_3 and need to be replaced. After the appropriate substitutions we obtain that the generating rays $r_1, \dots, r_{14}$

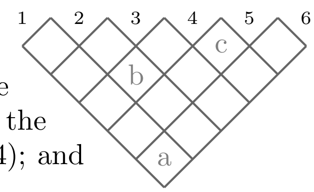
of $C_{\alpha_1,\alpha_2,\alpha_3}$ corresponding to the free variables $a_1, a_3, a_5, a_6, b_1, b_4, b_5, b_6, b_7, c_1, c_3, c_4, c_6, c_7$ respectively are:

	a_1	a_2	a_3	a_4	a_5	a_6	a_7	b_1	b_2	b_3	b_4	b_5	b_6	b_7	c_1	c_2	c_3	c_4	c_5	c_6	c_7
r_1	1	0	0	0	0	0	1	0	0	1	0	0	0	0	0	0	0	0	1	0	0
r_2	0	0	1	0	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0
r_3	0	0	0	0	1	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
r_4	0	0	0	0	0	1	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0
r_5	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0
r_6	0	0	0	1	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0
r_7	0	1	0	1	0	0	0	0	0	1	0	1	0	0	0	1	0	0	0	0	0
r_8	0	1	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0
r_9	0	1	0	0	0	0	1	0	0	1	0	0	0	1	0	0	0	0	1	0	0
r_{10}	0	0	0	1	0	0	0	0	0	3	0	0	0	0	1	0	0	0	0	0	0
r_{11}	0	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	1	0	0	0	0
r_{12}	0	1	0	0	0	0	1	0	0	1	0	0	0	0	0	0	0	1	0	0	0
r_{13}	0	0	0	0	0	0	1	0	0	1	0	0	0	0	0	0	0	0	0	1	0
r_{14}	0	0	0	0	0	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	1.

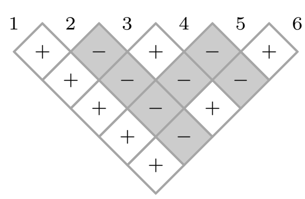
Appendix A: Sign diagrams

This appendix is devoted to *sign diagrams*, a method of displaying type A inversion sets which in some sense extends to complete flag varieties the Young diagrams used when describing Schubert cycles on Grassmanians. Although the use of sign diagrams is not necessary for the proofs of the theorems, many of our arguments have been guided by diagrammatic thinking and their point of view makes several statements in the paper transparent.

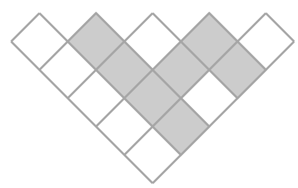
A.1. **Basic definition.** In order to display the inversion set of an element $\alpha \in S_n$ we start by listing the numbers 1,..., n across the page, and draw a triangular grid of squares below them, as illustrated at right in the case $n = 6$. Every square in the grid corresponds to exactly one (i, j) with $1 \leqslant i < j \leqslant n$; the square corresponding to (i, j) is the unique square which is directly southeast of i and directly southwest of j. In the picture we have labelled the sample squares (a) (1,6); (b) (2,4); and (c) (4,5).



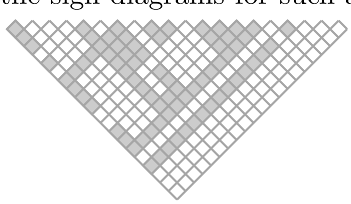
Given α we then mark all the squares corresponding to $(i, j) \in \Phi(\alpha)$ with a shaded "$-$" (to indicate that the positive root (i, j) is sent to a negative root by α), and mark those $(i, j) \notin \Phi(\alpha)$ with an unshaded "+" (to indicate that (i, j) is sent to a positive root by α). In order to reduce clutter in the diagram we sometimes simply omit the +/− signs or the numbers 1,..., n at the top, since these may be deduced from the size and shading of the diagram. Here is the sign diagram for the inversion set of $\alpha = (1, 6, 3, 5, 2, 4) \in S_6$ displayed using the two different conventions.



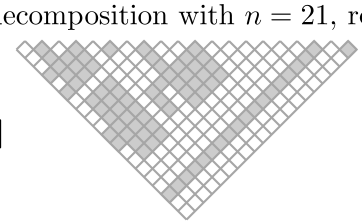
vs



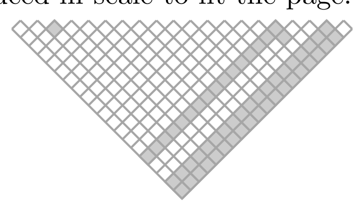
The main problem motivating the paper is describing decompositions of Δ_n^+. Here are the sign diagrams for such a decomposition with $n = 21$, reduced in scale to fit the page.



$\sqcup$



$\sqcup$

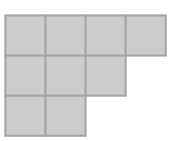


For large n the inversion sets can become quite intricate, revealing patterns reminicent of cellular automata.

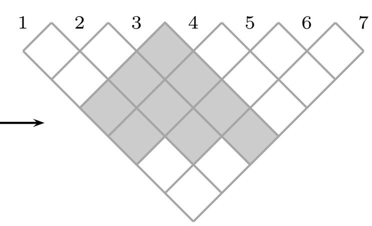
A.2. **Connection with Young diagrams.** Let $G(r,n)$ denote the Grassmanian of r-planes through the origin in $\mathbb{C}^n$ (with $1 \leqslant r \leqslant n$). The cohomology ring of $G(r,n)$ has a $\mathbb{Z}$-basis consisting of *Schubert cycles*: cohomology classes Poincaré dual to particular Zariski-closed subsets of $G(r,n)$. Fixing a complete flag in $\mathbb{C}^n$ (equivalently a Borel subgroup B of $\mathrm{GL}_n(\mathbb{C})$), the subsets are the closures of the points in $G(r,n)$ parameterizing those r-planes intersecting the elements of the flag in fixed dimensions (equivalently the closures of the B-orbits). The combinatorial object parameterizing the data of how the r-planes meet the fixed flag, and therefore parameterizing the cohomology classes, are the Young diagrams which fit into an $r \times (n-r)$ box.

A similar construction works for the variety $X = \mathrm{GL}_n(\mathbb{C})/B$ parameterizing complete flags in $\mathbb{C}^n$. Here the subsets are the Zariski closures of the set of points in X where the elements of the flag meet elements of the fixed flag in prescribed dimensions, or equivalently, the B-orbits on X. The combinatorial objects parameterizing the B-orbits in this case are the elements of S_n, the Weyl group of $\mathrm{GL}_n(\mathbb{C})$.

The Grassmanian $G(r,n)$ may be realized as $\mathrm{GL}_n(\mathbb{C})/P$, where P is a maximal parabolic subgroup containing B (which maximal subgroup depends on the value of r). We therefore have a quotient map $\pi\colon X \longrightarrow G(r,n)$, and this gives rise to the following procedure. Start with a Young diagram λ fitting in an $r \times (n-r)$ box, take the corresponding Schubert class $[\Sigma_\lambda]$ on $G(r,n)$, pull this back via π to a cohomology class $[\Sigma_\alpha]$ on X (with $\alpha \in S_n$), and finally take the inversion set of α, as represented by a sign diagram. Skipping the cohomology classes and showing only the combinatorial objects (Young diagram, element of S_n, and inversion set) here is an example from the cohomology of $G(3,7)$:

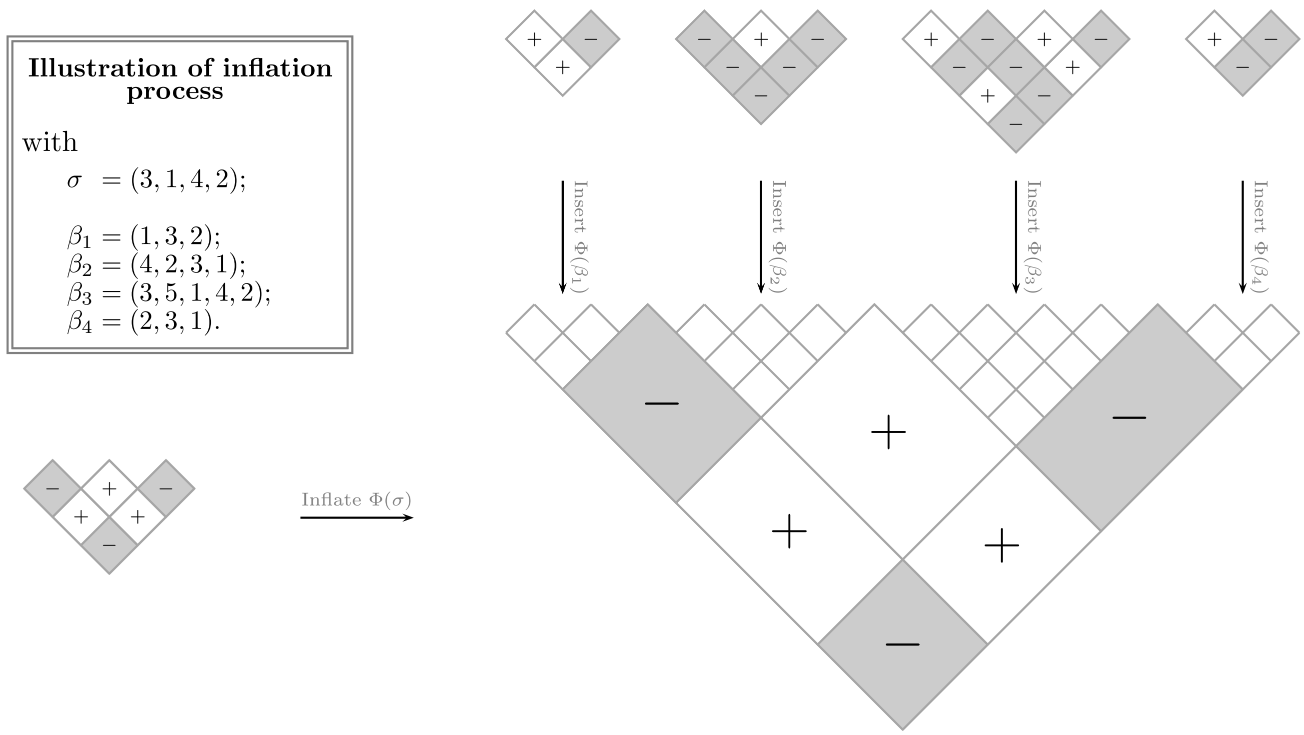


$$\longmapsto \qquad \alpha = (3,5,7,1,2,4,6) \qquad \longmapsto$$



The conclusion suggested by this example holds in general: the inversion set associated to a Young diagram λ by this procedure *is* that same Young diagram, rotated 45°. For a class on $G(r,n)$ the top corner of the Young diagram appears between the labels r and $r+1$.

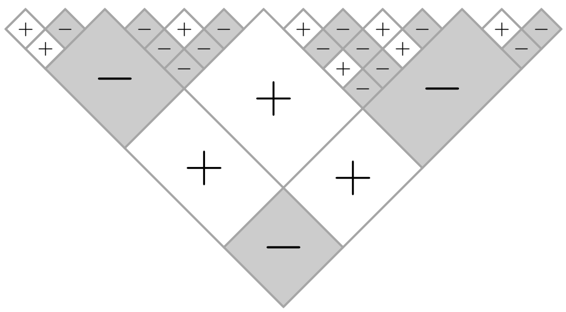
A.3. **Inflation.** The graphical description of inflation follows easily from the "shuffling cards" model. It is again easiest to explain with an example.



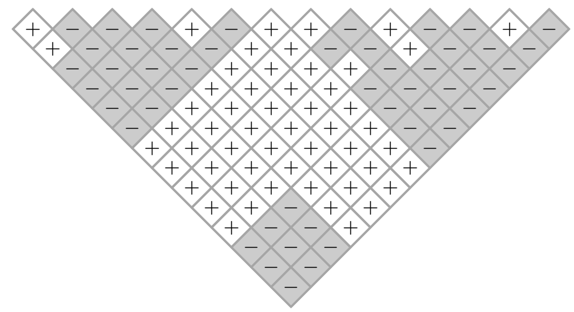
In this example the fact that $\beta_1, \ldots, \beta_4$ are elements of S_3, S_4, S_5, and S_3 respectively tells us that the resulting inflation is an element of S_n with $n = 3 + 4 + 5 + 3 = 15$, and that we should divide $\{1, \ldots, 15\}$ into the consecutive subsets $U_1 = \{1, 2, 3\}$, $U_2 = \{4, 5, 6, 7\}$, $U_3 = \{8, 9, 10, 11, 12\}$, and $U_4 = \{13, 14, 15\}$ of lengths 3, 4, 5, and 3 respectively.

The large blocks of $+$ and $-$ signs (indicated by the large blocks with a single $+$ or $-$) result from permuting the subsets $U_1, \ldots, U_4$ as prescribed by $\sigma \in S_4$. Explicitly, setting $\alpha = \sigma[\beta_1, \beta_2, \beta_3, \beta_4]$, for every $(i, j) \in \Phi(\sigma)$, we have $(a, b) \in \Phi(\alpha)$ for all $a \in U_i$, $b \in U_j$, and similarly for $(i, j) \notin \Phi(\sigma)$. Each element (i, j) of $\Phi(\sigma)$ therefore inflates to give an $|U_i| \times |U_j|$ block in $\Phi(\alpha)$ (length $|U_i|$ in the northeast-southwest direction, $|U_j|$ in the northwest-southeast direction). For each $(a, b) \in \Delta_n^+$ with a and b in different intervals, we thus know whether (a, b) is in $\Phi(\alpha)$ or not. However, as part of inflation we also permute each U_i using β_i, and this tells us how to decide on the status of those (a, b) with a, b in the same interval. Visually this amounts to simply inserting the sign diagram for $\Phi(\beta_i)$ in the appropriate empty space left by the inflation process. This procedure is the graphical translation of Lemma 3.2.

After inflating, we may leave the large blocks in the diagram to remind us of the inflation, or subdivide them into the usual smaller squares, depending on the situation. Thus the inflation above may be represented (again reduced in scale to fit the page) by



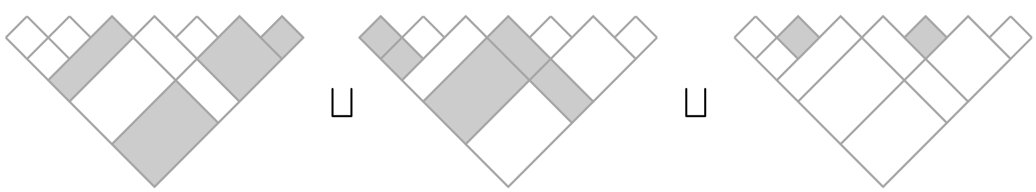
or



A.4. **Relation with ideas from the text.** In this subsection we use sign diagrams to illustrate some of the ideas from the main article.

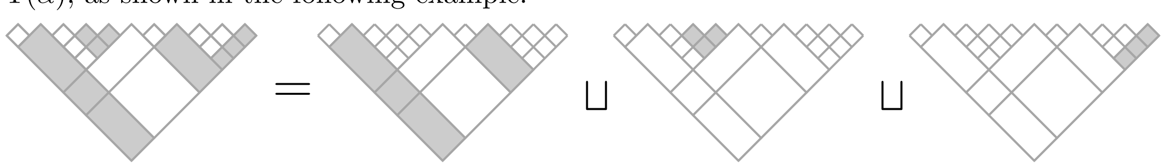
If σ's and β's give a decomposition, so do the inflations. As in §1.4, suppose that we divide $\{1, \ldots, n\}$ into m consecutive intervals $U_1, \ldots, U_m$, choose $\sigma_i \in S_m$, $i = 1, \ldots, r$ such that $\Delta_m^+ = \sqcup_i \Phi(\sigma_i)$, and furthermore choose $\beta_{ij} \in S_{|U_j|}$ for $i = 1, \ldots, r$, $j = 1, \ldots, m$ such that $\Delta_{|U_j|}^+ = \sqcup_i \Phi(\beta_{ij})$ for each j. Then it should be clear from the visual description of the inflation procedure that this implies the decomposition $\Delta_n^+ = \sqcup_i \sigma_i[\beta_{i1}, \ldots, \beta_{im}]$.

As an exercise the decomposition from Example 1.12, which is constructed in such a manner, is pictured below. The reader is invited to identify the diagrams inflated and inserted in each of the three pieces of the decomposition and check that they satisfy the hypotheses above.



Theorem 1.11, the central result of the paper, shows conversely that *every* decomposition of Δ_n^+ admits a recursive description by inflations satisfying the above conditions. The result of the theorem is more precise, identifying a canonical such description satisfying additional properties well suited to recursive analysis.

Rules for indecomposibility. If $\alpha = \sigma[\beta_1, \ldots, \beta_m]$ with each $\beta_i \in S_{z_i}$, then it is clear from the graphical procedure for inflation that we may use this description to decompose $\Phi(\alpha)$, as shown in the following example.

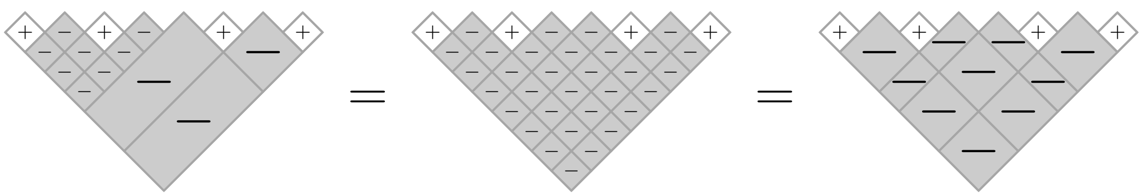


In formulas this kind of decomposition is written

$$\Phi(\alpha) = \Phi(\sigma[I_{z_1}, \ldots, I_{z_m}]) \sqcup \left(\bigsqcup_i \Phi(I_m[I_{z_1}, \ldots, I_{z_{i-1}}, \beta_i, I_{z_{i+1}}, \ldots, I_{z_m}]) \right).$$

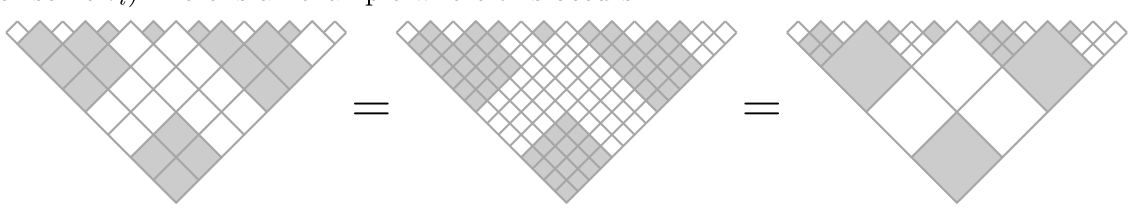
For the element α to be irreducible it follows from the inflation decomposition that at most one of σ, β_1,…, β_m can be different from the identity, and that this nontrivial element must itself be irreducible. This is the content of Corollary 3.5.

Rules for uniqueness in inflations. The sign diagram for J_m consists entirely of minus signs. If α is of the form $\alpha = J_m[\beta_1, \ldots, \beta_m]$ then of course these minus signs are inflated when making the sign diagram of α, and surround the sign diagrams of β_1,…, β_m. If some β_j also has this form (i.e., $\beta_j = J_{m'}[\tau_1, \ldots, \tau_{m'}]$ for some m') then some of the minus signs from $\Phi(\beta_j)$ may be merged with the minus signs from the inflation, as in the following example.



There is an identical problem (with the roles of the + and - signs reversed) for permutations of the form $\alpha = I_m[\beta_1, \ldots, \beta_m]$, where some β_j is also of the form $\beta_j = I_{m'}[\tau_1, \ldots, \tau_{m'}]$. In such cases we obtain uniqueness of the representation as an inflation by requiring that the diagram of J_m or I_m which is inflated account for as many of the $-$ or $+$ signs in $\Phi(\alpha)$ as possible (i.e., that m be as large as possible). In the example considered the diagram on the right is the one corresponding to the maximal J_m, with $m = 5$.

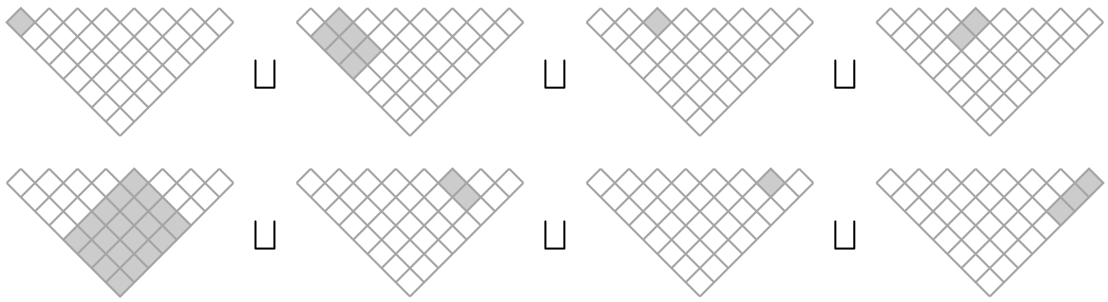
Somewhat the opposite problem occurs for representations of the form $\alpha = \sigma[\beta_1, \ldots, \beta_m]$ with $\sigma \neq I_m, J_m$. In this case it may be that σ can itself be represented in a non-trivial way as an inflation, and this description can then be propagated upwards to give a different representation of α as an inflation (i.e., if $\sigma = \gamma[\delta_1, \ldots, \delta_r]$ then we may write $\alpha = \gamma[\tau_1, \ldots, \tau_s]$ for some τ_i). Here is an example where this occurs.



In these cases we obtain uniqueness of the representation by requiring that σ be simple. This amounts to looking for $\sigma \in S_m$ with m as small as possible. In the example considered the diagram on the right is the one with smallest m, with $m = 4$. The two goals (m as large as possible and m as small as possible) seem to be in opposition, however, as Theorem 1.8

guarantees, every α has a unique representation as in inflation $\alpha = \sigma[\beta_1, \ldots, \beta_m]$ with either σ simple with $m \geqslant 4$ or σ one of I_m or J_m and m as large as possible.

Recursion for type A maximal decompositions. In §6 we considered the problem of enumerating the decompositions of Δ_n^+ of maximal length, i.e., into a decomposition of $n-1$ nonempty inversion sets. Here is a picture of such a decomposition with $n = 8$.



The example is relatively small, but is enough to infer the general structure of the problem.

The key is the diagram containing the highest root $(1, n)$ (i.e, the bottom vertex of the triangle), which in the example is the diagram at lower left. Because the inversion set also contains exactly one simple root it follows that it must consist of the entire rectangle with corners $(1, n)$ and that simple root. To see why we look at the example. In the diagram at lower left the only simple root inverted is $(5, 6)$. This means that the numbers $\{1, 2, 3, 4, 5\}$ all retain their relative order when α is applied, and that the same holds for $\{6, 7, 8, 9\}$. Combined with the fact that the inversion set contains $(1, 9)$, so that $\alpha(9) < \alpha(1)$, we deduce that α swaps the two intervals, i.e., that $\alpha = (6, 7, 8, 9, 1, 2, 3, 4, 5)$, and therefore that $\Phi(\alpha)$ is the rectangle with corners $(5, 6)$ and $(1, 9)$.

Returning to the general case, removing the rectangle containing the highest root disconnects the diagram into two smaller diagrams, each of which must be filled in by the other parts of the decomposition. The number of maximal decompositions of each of these smaller rectangles may be computed inductively. Thus if we organize the counting of the number of maximal decompositions of Δ_{n+1}^+ by the rectangle containing the highest root, we immediately arrive at the recursive relation $\mathrm{Cat}_{\mathrm{A}}(n) = \sum_{k=1}^{n} \mathrm{Cat}_{\mathrm{A}}(k-1)\mathrm{Cat}_{\mathrm{A}}(n-k)$. This leads quickly to the result that the enumerative problem is solved by the Catalan numbers.

By induction one also deduces that every diagram in a maximal decomposition is a rectangle. In the example all but two of these rectangles are reduced to lines or single squares, but this is simply because the example is small.

A.5. **Diagrams for types B and C.** For us the sign diagrams have been an extremely useful method of visualizing or discovering arguments in the type A case, and so it is natural to try and extend them to other types. Our method of displaying the type A inversion sets arose from picturing what Weyl group elements w do to an upper triangular Borel subgroup (this perspective has not been explained in the appendix), and one could try and repeat this idea in the other cases. However, in types B/C it turns out to be easier to use the group homomorphisms $\iota : \mathcal{W}(B_n) \hookrightarrow S_{2n+1}$ and $\iota : \mathcal{W}(C_n) \hookrightarrow S_{2n}$ from §5 and, rather than try and picture the inversion set of an element $\alpha \in \mathcal{W}(B_n) \cong \mathcal{W}(C_n)$ directly, to instead study the inversion set of $\iota(\alpha)$, the image of α under one of the homomorphisms. We first briefly recall the groups and the homomorphisms.

The Weyl groups $\mathcal{W}(B_n)$ and $\mathcal{W}(C_n)$ can be identified with the *signed permutations* of $\varepsilon_1,\ldots,\varepsilon_n$, i.e., we are allowed not only to permute the elements, but also multiply them by ± 1. Here is a sample element α of $\mathcal{W}(B_3) \cong \mathcal{W}(C_3)$:

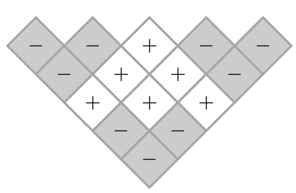
$$\alpha : \begin{cases} \varepsilon_1 \longrightarrow -\varepsilon_2 \\ \varepsilon_2 \longrightarrow \quad \varepsilon_3 \\ \varepsilon_3 \longrightarrow \quad \varepsilon_1 \end{cases} .$$

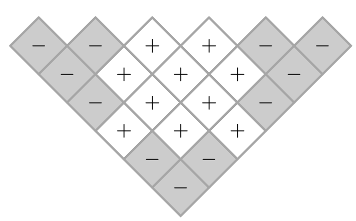
We can promote $\alpha \in \mathcal{W}(C_n)$ to an element $\iota(\alpha) \in S_{2n}$ by considering $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n, -\varepsilon_n, -\varepsilon_{n-1},\ldots, -\varepsilon_1$ to be distinct symbols, and using the rule given by α (and "linearity") to deduce a permutation of these $2n$ elements. For example, the element $\alpha \in \mathcal{W}(C_3)$ shown above corresponds to

$$\iota(\alpha) : \begin{cases} \varepsilon_1 \longrightarrow -\varepsilon_2 \\ \varepsilon_2 \longrightarrow \quad \varepsilon_3 \\ \varepsilon_3 \longrightarrow \quad \varepsilon_1 \\ -\varepsilon_3 \longrightarrow -\varepsilon_1 \\ -\varepsilon_2 \longrightarrow -\varepsilon_3 \\ -\varepsilon_1 \longrightarrow \quad \varepsilon_2 \end{cases} .$$

Using the order $\varepsilon_1, \varepsilon_2, \varepsilon_3, -\varepsilon_3, -\varepsilon_2, -\varepsilon_1$, this is the element $\iota(\alpha) = (5, 3, 1, 6, 4, 2) \in S_6$. We can similarly obtain an element in S_{2n+1} by adding the element 0 (in the order ε_1, $\ldots$, ε_n, 0, $-\varepsilon_n$, $\ldots$, $-\varepsilon_1$) and simply fixing 0. In the example considered, this gives the element $(6, 3, 1, 4, 7, 5, 2) \in S_7$. (The way we have presented this rule 'adding 0' seems somewhat arbitrary, but it does make sense from the natural description of the complete flag variety of B_n type as a subvariety of the complete flag variety of A_{2n} type.) We will use the symbol $\iota(\alpha)$ for the image of $\alpha \in \mathcal{W}(C_n) \cong \mathcal{W}(B_n)$ in S_{2n} or S_{2n+1} under either of these homomorphisms, and trust that the resulting permutation (of either an even or odd number of elements) will reveal which homomorphism was intended.

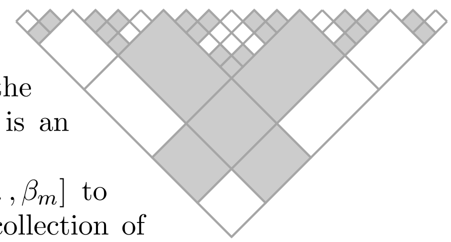
Here are the sign diagrams for $\iota(\alpha)$ (in S_6 and S_7) for the sample element of $\mathcal{W}(C_3)$ considered above.





The images of the injective homomorphisms $\mathcal{W}(C_n) \longrightarrow S_{2n}$ and $\mathcal{W}(B_n) \longrightarrow S_{2n+1}$ turn out to be precisely those elements whose sign diagram is symmetric about the vertical centre line, and the basic idea is to simply study such 'symmetric' sign diagrams and the corresponding elements of S_{2n} and S_{2n+1}.

One point is worth stating explicitly: given $\alpha \in \mathcal{W}(C_n)$ or $\mathcal{W}(B_n)$, the inversion set $\Phi(\alpha)$ is a subset of $\Delta^+_{C_n}$ (or $\Delta^+_{B_n}$), while the inversion set $\Phi(\iota(\alpha))$ is a subset of Δ^+_{2n} or Δ^+_{2n+1}, and these sets can be quite different. For instance these sets almost never have the same number of elements. (This is evident in the example above, where inversion sets for $\iota(\alpha)$ in S_6 and S_7 don't have the same number of elements as each other and so could not both agree with the number of elements in $\Phi(\alpha)$.) More importantly, the ideas of "indecomposable", "decomposition", "disjoint", or "simple" could potentially be quite different in $\Delta^+_{C_n}$ and Δ^+_{2n} (or $\Delta^+_{B_n}$ and Δ^+_{2n+1}), and one of the main things we need to check is that in fact they are not.



In order to even define "simple" we need to have a notion of inflation, and to do this we simply use the inflation procedure in S_{2n} or S_{2n+1} but require that all the data describing the inflation be 'symmetric'. At right is an example.

For the data describing an inflation $\iota(\alpha) = \sigma[\beta_1, \ldots, \beta_m]$ to be symmetric means that: (i) σ is symmetric; (ii) the collection of intervals $U_1, \ldots, U_m$ are symmetric (i.e, interchanged by the operation of reversing $1, \ldots, 2n$ or $1, \ldots, 2n+1$); (iii) if there is an interval U_j which is itself symmetric (i.e., straddles the centre line) then the corresponding β_j must be symmetric; and (iv) for all other intervals U_j the sign diagram for β_j must be the mirror image of the sign diagram for β_{m+1-j}. The example presented above has all these features. Conditions (ii), (iii) and (iv) may be summarized by the condition that $\beta_{m+1-j} = J_{z_j}\beta_j J_{z_j}$ for $j = 1, \ldots, m$.

Propositions 4.1 and 5.1 contain the useful result that if a symmetric $\iota(\alpha)$ can be represented nontrivially as an inflation, it can be represented nontrivially as an inflation with symmetric data. With the idea of inflation in place, we now define an element $\alpha \in \mathcal{W}(B_n)$ or $\mathcal{W}(C_n)$ to be simple if the corresponding $\iota(\alpha)$ is simple in S_{2n} or S_{2n+1}.

In §5 the following results are established showing that the intrinsic notions for an inversion set $\Phi(\alpha)$ with $\alpha \in \mathcal{W}(C_n) \cong \mathcal{W}(B_n)$ in type B/C agree with the the type A notions of the corresponding element $\iota(\alpha)$ in S_{2n} or S_{2n+1}.

(i) Corollary 5.2: α is irreducible if and only if $\iota(\alpha)$ is irreducible; $\Phi(\alpha_1)$ and $\Phi(\alpha_2)$ are disjoint if and only if $\Phi(\iota(\alpha_1))$ and $\Phi(\iota(\alpha_2))$ are disjoint; $\Delta^+_{B_n} = \sqcup_i \Phi(\alpha_i)$ if and only if $\Delta^+_{2n+1} = \sqcup_i \Phi(\iota(\alpha_i))$ (respectively $\Delta^+_{C_n} = \sqcup_i \Phi(\alpha_i)$ if and only if $\Delta^+_{2n} = \sqcup_i \Phi(\iota(\alpha_i))$).

(ii) Proposition 5.3: α is simple if and only if $\iota(\alpha)$ is simple.

With these results, one deduces Theorem 5.4 which is the type B/C version of Theorem 1.11. The arguments and pictures used in §A.4 also extend in an appropriate way to the B/C case. For instance, one may also deduce a uniqueness statement for representation as a symmetric inflation, paralleling that of Theorem 1.8, or recursion relations for the type B/C Catalan numbers (Proposition 6.2).

10. Acknowledgements

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