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RESEARCH ARTICLE

Seshadri constants on $\mathbb{P}^1 \times \mathbb{P}^1$ and applications to the symplectic packing problem

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Abstract
In this paper we compute the r-point Seshadri constant on $\mathbb{P}^1 \times \mathbb{P}^1$ for those line bundles where the answer might be expected to be governed by (-1)-curves. As a consequence we obtain explicit formulas for the symplectic packing problem for $\mathbb{P}^1 \times \mathbb{P}^1$. Some exact values for the r-point Seshadri constant outside the region governed by Mori's cone theorem are also given. These latter results use a useful new "reflection method".

In the analysis there is a striking difference between the cases when r is odd and when r is even. When r is even the problem admits an infinite order automorphism, and there are infinitely many (-1)-curves to consider. In contrast, when r is odd only a finite number (usually four) types of (-1)-curves are relevant to our answer.

Contents

1. Introduction

1.1. r-point Seshadri constants

Let Y be a smooth projective surface and $\pi\colon X \longrightarrow Y$ the blowup of Y at r very general points $p_1, \dots,$ $p_r \in Y$. We denote by $E_1, \dots, E_r$ the exceptional divisors of π, with E_i lying over p_i, and use $E = \sum_{i=1}^{r} E_i$ for their sum. Given an ample line bundle L on Y, the *r-point Seshadri constant* of L is defined to be

$$\epsilon_r(L) := \sup\Big\{\gamma \geqslant 0 \mid \pi^* L - \gamma E \text{ is nef}\Big\}. \tag{1.1}$$

Equivalently

$$\epsilon_r(L) = \inf_C \left\{\frac{C \cdot \pi^* L}{C \cdot E}\right\}, \tag{1.2}$$

where the infimum is over the effective curves C in X which do not contain any E_i as a component. We adopt the convention that C does not have to intersect any of the E_i. In such cases $(C \cdot \pi^* L)/(C \cdot E)$, interpreted as $+\infty$, does not affect the infimum, and this convention allows us to avoid many repetitions of "for those C which also intersect at least one of the E_i".

To our knowledge r-point Seshadri constants were first introduced by Küchle [7], for smooth projective varieties Y of arbitrary dimension. In general very few exact values of $\epsilon_r(L)$ are known. For instance,

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when $Y = \mathbb{P}^2$, computing $\epsilon_r(\mathcal{O}_{\mathbb{P}^2}(1))$ is equivalent to the Nagata conjecture, a problem which is open for all $r \geqslant 10$ where r not a square.

In this paper we restrict to the surface $Y = \mathbb{P}^1 \times \mathbb{P}^1$. We use $L = \mathcal{O}_Y(e_1, e_2)$ for the line bundle on $\mathbb{P}^1 \times \mathbb{P}^1$ of bidegree (e_1, e_2). Such a line bundle L is nef if and only if $e_1, e_2 \geqslant 0$, and ample if and only if $e_1, e_2 \geqslant 1$. By the *slope* of L we mean e_2/e_1, allowing ∞ if $e_1 = 0$ and $e_2 \neq 0$.

1.2. *Definition of α_r and β_r*

For a positive integer r we set

$$\alpha_r := \frac{(r-4) + \sqrt{r(r-8)}}{4} \quad \text{and} \quad \beta_r := \frac{(r-4) - \sqrt{r(r-8)}}{4}.$$

The numbers α_r and β_r are the roots of $t^2 - ((r-4)/2)t + 1 = 0$. When r is even α_r and β_r are mutually inverse units in the ring of integers of $\mathbb{Q}[\sqrt{r(r-8)}]$. When $r \geqslant 10$ this ring is a real quadratic extension of $\mathbb{Q}$, and α_r and β_r are of infinite order. The numbers α_r and β_r govern the problem of computing $\epsilon_r(L)$ on $\mathbb{P}^1 \times \mathbb{P}^1$ in several ways. Here is the first.

1.3. *Inner and outer bundles*

We call a nef bundle $L = \mathcal{O}_Y(e_1, e_2)$ an *inner bundle* if $\frac{e_2}{e_1} \in [\beta_r, \alpha_r]$, and an *outer bundle* otherwise. The motivation for this terminology comes from Figure 1 on page 10. We note that whether any particular L is an inner or outer bundle depends on the value of r.

Let $p_1, \ldots, p_r \in Y$ be very general points, and $\pi\colon X \longrightarrow Y$ the blowup of Y at $p_1, \ldots, p_r$ as in §1.1. For a line bundle L on $\mathbb{P}^1 \times \mathbb{P}^1$, and any $\gamma \geqslant 0$ we set

$$L_\gamma := \pi^* L - \gamma \sum_{i=1}^{r} E_i = \pi^* L - \gamma E. \tag{1.3}$$

By definition of the r-point Seshadri constant, if $\gamma > \epsilon_r(L)$ then the class L_γ is not nef, and therefore there is an irreducible curve $C \subset X$ such that $L_\gamma \cdot C < 0$. In §2.4 we give a heuristic argument that if L is an outer bundle, then one might expect that C is K_X-negative. Thus by the fact that in such a case one must also have $C^2 < 0$, it would follow that C is a (-1)-curve.

One consequence of our analysis in the paper is that this guess is correct, and we are able to explicitly compute $\epsilon_r(L)$ for all outer bundles and all r. The answer appears in §1.5 after discussing (in §1.4) another appearance of α_r and β_r.

A symmetrization procedure. Set F_1 and F_2 to be the pullback to X of the fibre classes $\mathcal{O}_Y(1,0)$ and $\mathcal{O}_Y(0,1)$ respectively, and let $V_r \subset H^2(X, \mathbb{R})$ be the subspace of the real Néron-Severi group spanned by F_1, F_2, and E. Thus V_r is a three-dimensional real vector space, and for vectors $v = d_1 F_1 + d_2 F_2 - mE$ and $w = e_1 F_1 + e_2 F_2 - nE$ in V_r the intersection pairing between v and w is given by

$$v \cdot w = d_1 e_2 + d_2 e_1 - mnr. \tag{1.4}$$

By (1.3) the class of L_γ is in V_r. If L_γ is not nef the following argument shows that there is always an effective curve C with class in V_r such that $L_\gamma \cdot C < 0$.

Let C' be an irreducible curve such that $L_\gamma \cdot C' < 0$, and let $d_1 F_1 + d_2 F_2 - \sum_{i=1}^r m_i E_i$ be the class of C'. Since the points $p_1, \ldots, p_r$ are general it follows that for any permutation σ on $\{1, \ldots, r\}$ there is an irreducible curve C'_σ with class $d_1 F_2 + d_2 F_2 - \sum_{i=1}^r m_{\sigma(i)} E_i$. Moreover, $L_\gamma \cdot C' = L_\gamma \cdot C'_\sigma$.

Let σ be an r-cycle, and set C to be the sum $C := \sum_{i=1}^r C'_{\sigma^i}$. Then C is an effective curve, of class $r d_1 F_1 + r d_1 F_2 - \left(\sum_{i=1}^r m_i\right) E \in V_r$, and $L_\gamma \cdot C = r(L_\gamma \cdot C') < 0$.

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More generally this symmetrization argument shows that the restriction of the nef and effective cones to V_r are cones which are still dual in V_r. Thus, to understand $\epsilon_r(L)$ we may restrict our attention to V_r.

1.4. Automorphisms of the problem for even r

It is easy to verify that the linear transformation $T_r : V_r \longrightarrow V_r$ given, in the basis F_1, F_2, and E, by the matrix

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & \frac{r}{2} & r \\ 0 & -1 & -1 \end{bmatrix} \tag{1.5}$$

preserves the intersection form. When r is even we show in Theorem 3.1 that T_r is an automorphism of the problem, in the sense that if $\xi \in V_r$ is any class, then ξ is nef, or effective, or represents a curve with s irreducible components, if and only if $T_r(\xi)$ is respectively nef, effective, or represents a curve with s irreducible components. These statements are not true when r is odd.

The transformation T_r has eigenvalues α_r, β_r, and 1, with respective eigenvectors (in coordinates given by F_1, F_2, and E)

$$v_{\alpha_r} = \left(\tfrac{1}{\alpha_r+1}, \tfrac{\alpha_r}{\alpha_r+1}, -\tfrac{2}{r}\right), \quad v_{\beta_r} = \left(\tfrac{\alpha_r}{\alpha_r+1}, \tfrac{1}{\alpha_r+1}, -\tfrac{2}{r}\right), \quad \text{and } v_1 = (-2, -2, 1). \tag{1.6}$$

We note that v_1 is the class of K_X, that v_{α_r} and v_{β_r} are exchanged by the automorphism exchanging F_1 and F_2. Additionally, since $\alpha_r\beta_r = 1$, v_{β_r} may also be written as $v_{\beta_r} = \left(\frac{1}{\beta_r+1}, \frac{\beta_r}{\beta_r+1}, -\frac{2}{r}\right)$. Thus the Galois automorphism of $\mathbb{Q}[\sqrt{r(r-8)}]$ exchanging α_r and β_r exchanges v_{α_r} and v_{β_r}, although we will not use this fact.

If $r \geqslant 10$ then α_r and β_r are units of infinite order, and thus T_r also has infinite order. Starting with a nef or effective class and iterating T_r then allows us to produce infinitely many other nef or effective classes. This is the key to our computation of $\epsilon_r(L)$ for even r and outer bundles L.

When $r \geqslant 10$ we have $0 < \beta_r < 1 < \alpha_r$. Thus, in forward iterations of T_r vectors generally converge (modulo scaling) to v_{α_r}, and under backwards iterations to v_{β_r}. As a consequence if r is even then both v_{α_r} and v_{β_r} are limits of nef classes, and are therefore also nef. They are also *square-zero classes*, $v_{\alpha_r}^2 = 0 = v_{\beta_r}^2$, and so on the boundary of the nef cone.

A function on the square zero cone. Before proceeding to the results for $\epsilon_r(L)$ we make two more digressions. For a nef line bundle $L = \mathcal{O}_Y(e_1, e_2)$ we define the *numerical bound*, $\eta_r(L)$ by

$$\eta_r(L) := \sqrt{\tfrac{L^2}{r}} = \sqrt{\tfrac{2e_1e_2}{r}}. \tag{1.7}$$

The value $\eta_r(L)$ is precisely the value of γ so that $L_\gamma^2 = 0$. In other words, $\eta_r(L)$ is the value of γ which puts L_γ on the cone of square-zero classes. The number $\eta_r(L)$ is therefore also an upper bound for the Seshadri constant: $\epsilon_r(L) \leqslant \eta_r(L)$.

For a vector $v \in V_r$ with $v^2 = 0$, and not a multiple of v_{α_r} or v_{β_r}, we put

$$\varphi_r(v) := \frac{\log\left(\frac{v \cdot v_{\beta_r}}{v \cdot v_{\alpha_r}}\right)}{2\log(\alpha_r)}. \tag{1.8}$$

This formula is justified by the following properties (see §6.2). For such a vector v, $\varphi_r(\lambda v) = \varphi_r(v)$ for any $\lambda \in \mathbb{R}$, $\lambda \neq 0$; $\varphi_r(T_r^n(v)) = \varphi_r(v) + n$ for all $n \in \mathbb{Z}$; and if $\tilde{v}$ is the vector obtained from v by the automorphism switching F_1 and F_2, then $\varphi_r(\tilde{v}) = -\varphi_r(v)$. Thus, φ_r is a map from the square-zero

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cone (up to scaling, and minus the lines spanned by v_{α_r} and v_{β_r}) to $\mathbb{R}$ which takes symmetries of the problem to similar symmetries on $\mathbb{R}$.

1.5. Seshadri constants for outer bundles

Here we concentrate on the cases $r \geqslant 9$. When $r \leqslant 7$ the blowup of $\mathbb{P}^1 \times \mathbb{P}^1$ at r general points is Fano, and the answers in those cases have a different character than the general case. In addition, one minor aspect of our description below is not valid for $r = 8$. These cases are discussed in §5.

In order to describe the answers in the even case, here and in the symplectic packing problem, it will be convenient to define several sequences $\{s_{n,r}\}_{n\in\mathbb{Z}}$ by giving the terms $s_{-1,r}$, $s_{0,r}$, and $s_{1,r}$, and defining all other terms by the recursion equation coming from the characteristic polynomial of T_r:

$$s_{n,r} = \tfrac{r-2}{2}\Big(s_{n-1,r} - s_{n-2,r}\Big) + s_{n-3,r}. \tag{1.9}$$

We define the sequence $\{p_{n,r}\}_{n\in\mathbb{Z}}$ by $p_{-1,r} = 0$, $p_{0,r} = 0$, $p_{1,r} = r$, and determine all other $p_{n,r}$ by the recursion (1.9). Similarly we define $\{m_{n,r}\}_{n\in\mathbb{Z}}$ by $m_{-1,r} = 1$, $m_{0,r} = -1$, $m_{1,r} = 1$, and (1.9). We note that $m_{n,r} = m_{-n,r}$ and $p_{n,r} = p_{-1-n,r}$ for all $n \in \mathbb{Z}$.

Theorem 1.1 (Seshadri Constants for Outer Bundles). *Suppose that $L = \mathcal{O}_Y(e_1, e_2)$ with $e_1, e_2 \geqslant 1$, and that L is an outer bundle, that is, that $\frac{e_2}{e_1} \notin [\beta_r, \alpha_r]$.*

***If r is odd, r ⩾ 9**. Then*

$$\epsilon_r(L) = \begin{cases} e_2 & \text{if } \frac{e_2}{e_1} \leqslant \frac{2}{r+1}, \\ \frac{2e_1+(r-1)e_2}{2r} & \text{if } \frac{e_2}{e_1} \in [\frac{2}{r+1}, \beta_r], \\ \frac{(r-1)e_1+2e_2}{2r} & \text{if } \frac{e_2}{e_1} \in [\alpha_r, \frac{r+1}{2}], \\ e_1 & \text{if } \frac{r+1}{2} \leqslant \frac{e_2}{e_1}. \end{cases} \tag{1.10}$$

***If r is even, r ⩾ 10**. Set $v_L = (e_1, e_2, -\eta_r(L)) \in V_r$, that is, set v_L to be the class of L_γ with $\gamma = \eta_r(L)$, and put $n = \lfloor \varphi_r(v_L) + \frac{1}{2} \rfloor$, where $\lfloor x \rfloor$ denotes the largest integer $\leqslant x$. Then*

$$\epsilon_r(L) = \frac{e_1 p_{n,r} + e_2 p_{n-1,r}}{r m_{n,r}}. \tag{1.11}$$

The explanation for (1.11) is as follows. Consider the classes $C_{n,r} = (p_{n-1,r}, p_{n,r}, -m_{n,r})$ defined by the sequences $\{p_{n,r}\}_{n\in\mathbb{Z}}$ and $\{m_{n,r}\}_{n\in\mathbb{Z}}$. Then $C_{0,r} = (0, 0, 1)$ is the class of E, and for all $n \in \mathbb{Z}$, $C_{n,r} = T_r^n(C_{0,r})$. In Theorem 3.3 we show that when r is even the $C_{n,r}$ generate the effective cone of curves in V_r whose slopes lie outside of $[\beta_r, \alpha_r]$. It follows that for an outer bundle L one of these curves determines $\epsilon_r(L)$. The formula with φ_r above is one possible method of locating the correct n, and intersecting with $C_{n,r}$ then gives (1.11).

For the proofs of the results in the even case see §3, and for the odd case, §4.

1.6. Applications to the symplectic packing problem

We recommend [10, §1] and [2] for a discussion of the history of this problem and the reasons for its interest. Here we give a brief outline oriented towards our application of the previous results to the symplectic packing problem for $\mathbb{P}^1 \times \mathbb{P}^1$.

Let (M, ω_M) be a closed symplectic manifold of real dimension 4, and let $B_\lambda \subset \mathbb{R}^4$ denote the ball of radius λ centred at 0, equipped with the restriction of the standard symplectic form on $\mathbb{R}^4$: $\omega_{\mathbb{R}^4} = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$.

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For a given r, consider all possible symplectic embeddings of r disjoint copies of B_λ into M, and denote by $\hat{\nu}_r(M)$ the supremum of the volumes which can be filled by such embeddings (i.e., the supremum of $r\pi^2\lambda^4$, over those λ for which there exists such a symplectic embedding of r disjoint B_λ into M). Finally, set $\nu_r(M) = \hat{\nu}_r(M)/\mathrm{Vol}(M)$, where the volume of M, like the volume of B_λ, is computed using the volume form $\omega_M \wedge \omega_M$.

Two basic questions are: (1) What is $\nu_r(M)$ for different values of r?; and (2) for which r does $\nu_r(M) = 1$? If $\nu_r(M) < 1$ one says that there is a *packing obstruction*, while if $\nu_r(M) = 1$ one says that there is a *full packing*.

Let Y be a smooth projective surface, and L a real ample class on Y. The first Chern class $c_1(L)$ can be represented by a Kähler form ω_L, which, when written out in terms of the underlying real coordinates, is a real symplectic form. We consider the packing problem for the real manifold M underlying Y, with symplectic form ω_L. To align our notation with the notation in the rest of the paper, we will use $\nu_r(L)$ for the value of $\nu_r(M)$ in this situation.

A remarkable discovery of [10] is that (-1)-curves in X provide obstructions to full packings. Even more striking is that this obstruction looks much like the Seshadri constant, except with the test curves C in (1.2) limited to (-1)-curves. To set this up we first extend definition (1.7) to any such pair (Y, L) by setting $\eta_r(L) = \sqrt{L^2/r} = \sqrt{\mathrm{Vol}(M)/r}$. Then we set

$$\tilde{\epsilon}_r(L) = \min\left(\inf_C\left\{\frac{C\cdot\pi^*L}{C\cdot E}\right\}, \eta_r(L)\right) \tag{1.12}$$

where this time the infimum is over irreducible (-1)-curves C in X distinct from the exceptional divisors E_i. Following our convention in §1.1, if there is no such (-1)-curve C which intersects any of the E_i the infimum is interpreted as ∞, and then $\tilde{\epsilon}_r(L) = \eta_r(L)$.

The obstruction result of [10] is that one always has $\nu_r(L) \leqslant \left(\frac{\tilde{\epsilon}_r(L)}{\eta_r(L)}\right)^2$. The paper is concerned with the case $Y = \mathbb{P}^2$, but the obstruction argument does not depend on this. Even more remarkably, a result of Biran, [1, Theorem 6.A], asserts that there is a class of surfaces, which includes $\mathbb{P}^2$ and ruled surfaces, where one has $\nu_r(L) = \left(\frac{\tilde{\epsilon}_r(L)}{\eta_r(L)}\right)^2$ for all L.

The Seshadri constants in Theorem 1.1 were obtained using (-1)-curves, or their symmetrized versions. As a result we can compute $\tilde{\epsilon}_r(L)$ and so, thanks to the result of Biran cited above, $\nu_r(L)$, for all real ample line bundles on $Y = \mathbb{P}^1\times\mathbb{P}^1$; equivalently, by [8, Theorem 1.1], for all symplectic forms on the underlying real manifold. As before we list the results for $r \geqslant 9$; the results for $r \leqslant 8$ appear in §7.2.

Theorem 1.2 (Formulas for the symplectic packing constant). *Let L be a real ample line bundle of type (e_1, e_2) (i.e., e_1 and e_2 are positive real numbers).*

If r is odd, $r \geqslant 9$. *Then*

$$\nu_r(L) = \begin{cases} \frac{re_2}{2e_1} & \text{if } \frac{e_2}{e_1} \leqslant \frac{2}{r+1}, \\ \frac{(2e_1+(r-1)e_2)^2}{8re_1e_2} & \text{if } \frac{e_2}{e_1} \in [\frac{2}{r+1}, \frac{2}{(\sqrt{r}-1)^2}], \\ 1 & \text{if } \frac{e_2}{e_1} \in [\frac{2}{(\sqrt{r}-1)^2}, \frac{(\sqrt{r}-1)^2}{2}], \\ \frac{((r-1)e_1+2e_2)^2}{8e_1e_2} & \text{if } \frac{e_2}{e_1} \in [\frac{(\sqrt{r}-1)^2}{2}, \frac{r+1}{2}], \\ \frac{re_1}{2e_2} & \text{if } \frac{r+1}{2} \leqslant \frac{e_2}{e_1}. \end{cases} \tag{1.13}$$

If r is even, $r \geqslant 10$. *Then*

$$\nu_r(L) = \begin{cases} 1 & \text{if } \frac{e_2}{e_1} \in [\beta_r, \alpha_r] \\ \frac{r(\epsilon_r(L))^2}{2e_1e_2} & \text{if } \frac{e_2}{e_1} \notin [\beta_r, \alpha_r] (\text{with } \epsilon_r(L) \text{ computed by the rule in } (1.11)). \end{cases} \tag{1.14}$$

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1.7. Conditions for full packings

Define sequences $\{q_{n,r}\}_{n\in\mathbb{Z}}$ by $q_{-1,r} = 1$, $q_{0,r} = 0$, $q_{1,r} = 1$, and the recursion (1.9). For this sequence one has $q_{n,r} = q_{-n,r}$ for all n. Taking into account the cases $r \leqslant 8$ (see §5), and reversing the formulae in Theorem 1.2, we get the following answer to question (2).

Theorem 1.3 (Conditions for full packings). ***If* r *is odd*.** *Then* $\nu_r(L) = 1$ *if and only if* $r \geqslant \max\left(\left(\sqrt{\frac{2e_2}{e_1}}+1\right)^2, \left(\sqrt{\frac{2e_1}{e_2}}+1\right)^2, 9\right)$.

***If* r *is even*.** *Then* $\nu_r(L) = 1$ *if and only if*

(i) $r \geqslant \dfrac{2(e_1+e_2)^2}{e_1e_2}$, *or*

(ii) *r is a value for which $\frac{e_2}{e_1}$ is equal to $\frac{q_{n+1,r}}{q_{n,r}}$ for some n.*

For a given (e_1, e_2), there is at most one value of r for which case (*ii*) occurs, see Theorem 6.4.

The lower bounds on r differ in the even and odd cases. Consequently by picking a line bundle with an extreme slope, we can find examples of line bundles L with ranges where full packings exist only for even r. Here are two examples with similar slopes.

Example 1.4. For the first, we give an example where case (*ii*) above does not occur. If $L = \mathcal{O}_Y(2, 401)$ then by Theorem 1.3 there is

- no full packing for any $r \leqslant 405$;
- a full packing for every $r \geqslant 443$; and
- for $r \in [406, 442]$ a full packing only for even r.

Example 1.5. Similarly, If $L = \mathcal{O}_Y(1, 200)$ then we find by Theorem 1.3 that there is

- no full packing for any $r \leqslant 399$;
- a full packing for every $r \geqslant 441$; and
- for $r \in [400, 440]$ a full packing only for even $r \geqslant 406$ and for $r = 400$.

In this second example $r = 400$ is an “unusual” r, that is, appears because of case (*ii*).

This phenomenon seems very surprising to the authors, even knowing the proofs of the formulas. For instance, returning to the first example, there is a full packing when $r = 410$. If we look for a packing with $r = 409$, then we could start with a packing for $r = 410$ and use 409 of the balls. Admittedly, that is not yet a full packing, but surely it would be possible to increase the radius and move the centres just a little bit to make up for it, and not have the balls intersect . . . ? Of course, the results above say that it is not possible. As a consistency check, Theorem 1.2 gives $\nu_{409}(L) = \frac{654481}{656036} \approx 0.9976297\ldots$, larger than the ratio $\frac{409}{410} \approx 0.99756097\ldots$ obtained using 409 out of the 410 balls but without increasing the radius.

The proofs for the above results on symplectic packing, using the previous results about Seshadri constants, appear in §7.

1.8. Results for inner bundles

One implication of the SHGH conjecture (see, for example, [3, §1.4]) and our analysis of which (−1)-curves affect Seshadri constants is that there should be a portion of the nef cone which is round. Specifically, for $r \geqslant 9$, we should have $\epsilon_r(L) = \nu_r(L)$ for all L whose slopes are in, respectively, $[\beta_r, \alpha_r]$ if r is even, and $[\frac{2}{(\sqrt{r}-1)^2}, \frac{(\sqrt{r}-1)^2}{2}]$ if r is odd.

If this description of the nef cone is correct, then the boundary of the nef cone, for slopes in the ranges indicated above, consists of classes ξ which are nef, square-zero (i.e., $\xi^2 = 0$) and K_X-positive: $K_X \cdot \xi > 0$. In this paper we call such classes inner square-zero nef classes.

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Finding such classes is quite useful. By definition, if ξ is nef, then there are no effective classes on the half plane $\xi^{<0}$. If ξ is K_X-positive, this half plane will contain a large proportion of K_X-positive classes, and it is exactly these classes whose existence we usually have the greatest difficulty in ruling out. In addition, if $\xi^2 = 0$ it means that ξ is on the boundary of the nef cone, and so provides the strongest condition on restricting effective classes.

One of the contributions of this paper is to construct such inner square-zero nef classes for all $r \geqslant 9$. To our knowledge, this is the first construction of such classes on the blowup of $\mathbb{P}^1 \times \mathbb{P}^1$ at r general points. These classes are constructed in §9, using a new "reflection method". If r is even we obtain, using T_r, infinitely many such classes, but if r is odd we only construct finitely many. In §10 we use pullback maps to construct other infinite families of such classes when r is even.

Given such a class ξ, say of the form $\xi = (e_1, e_2, -\sqrt{2e_1e_2/r})$ (the last coordinate is determined by the condition that $\xi^2 = 0$), then $\epsilon_r(L) = \eta_r(L)$ for the bundle $L = \mathcal{O}_Y(e_1, e_2)$. We are thus able to exhibit classes which achieve the predicted value of the Seshadri constant.

1.9. Relation with other work, I

Seshadri constants on $\mathbb{P}^1 \times \mathbb{P}^1$ and the related symplectic packing problem were studied in the 2005 Ph.D. thesis of W. Syzdek, the published version of which appears as [12].

In [12] Syzdek finds the same curves $C_{n,r}$ (from §1.5) which we use to compute the Seshadri constant in the even case. More precisely, our $C_{n,r}$ are each the disjoint union of $r(-1)$-curves (for instance $C_{n,r} = T_r^n(E)$, and E is the disjoint union of the r exceptional divisors), and Syzdek finds instead the classes of these (-1)-curves. Specifically, our curve $C_{n,r}$ with $n \geqslant 1$ is the symmetric orbit of the curve called $M_{D_{n+2}}$ in [12, Proposition 3.9], with $l = \frac{r-6}{2}$ and with the bidegrees switched. These curves ($C_{n,r}$, or its components D_{n+2}) impose the same conditions on the r-point Seshadri constants.

The approach in [12] cannot rule out that there may be other curves which affect the Seshadri constant when $\frac{e_2}{e_1} \notin [\beta_r, \alpha_r]$, and so for the corresponding line bundles can only give an upper bound on $\epsilon_r(L)$, and an upper bound on the symplectic packing number. In our argument we can conclude that these upper bounds are the actual values. The extra piece of information in our method is that we know that the iterates $T_r^n(F_2)$ are nef, and duality of the nef and effective cones then eliminates the possibility of other such curves.

In the case that r is odd, the (-1)-curves we find also already appear in [12, Table 4]. For instance, our curve of bidegree $(\frac{r-1}{2}, 1)$ is the curve of bidegree $(k+n+3, 1)$ in Table 4 when $r = 2k+2n+7$. Theorem 3.17 of [12] seems to claim, in the case r is odd, that the curves in [12, Table 4] compute the Seshadri constant for all (e_1, e_2) with $\frac{e_2}{e_1} \notin [\frac{2}{(\sqrt{r}-1)^2}, \frac{(\sqrt{r}-1)^2}{2}]$. We do not know how to justify this claim since we cannot rule out the possibility that there may be curves C with $C^2 < 0$ and $C \cdot K_X > 0$ which could impose a stronger condition on the Seshadri constant. In fact, we have had to do some work in §4.2–§4.3 to show that if such a curve exists, it at least could not affect line bundles with $\frac{e_2}{e_1} \notin [\beta_r, \alpha_r]$.

As part of question (2) in §1.6, one may ask for an r_0 so that for $r \geqslant r_0$ one has $\nu_r(L) = 1$. In general, as the examples in §1.7 suggest, it is the odd r which determine r_0. Our bound in Theorem 1.3 is sharp. Setting $s = \frac{e_2}{e_1}$, and assuming $s \geqslant 2$ to simplify the discussion, we get that there is a full packing for all r greater than $(\sqrt{2s}+1)^2 = 2s + 2\sqrt{2s} + 1$. In contrast, [12] (Definition 3.2, formula for R_0 with $a = 1$, $b = s$) gives a slightly worse estimate of

$$\frac{3+2s+3s^2}{2s} + \frac{(1+s)\sqrt{2(1+s^2)}}{s} = \left(\frac{3}{2} + \sqrt{2}\right)s + (1 + \tfrac{3}{\sqrt{2}}) + O\left(\frac{1}{s}\right) \quad \text{as } s \to \infty.$$

Finally, we should note that the dichotomy of behaviour between even and odd r, one aspect of the problem which we find surprising, already appears in [12]. For instance, in the estimates R_0 and r_0 in [12, Definition 3.2].

In summary, the improvements in this paper over the results of [12] are: (1) When r is even to give exact values of $\epsilon_r(L)$ for outer bundles L and exact values of $\nu_r(L)$ for all ample L; (2) When r is odd to justify the calculation of $\epsilon_r(L)$ for outer bundles L and to give the exact region where $\nu_r(L) = 1$. We are thus able give a complete answer to the symplectic packing problem for $\mathbb{P}^1 \times \mathbb{P}^1$. (3) To produce inner square-zero nef classes for all $r \geqslant 9$. Thus, to produce inner bundles where the Seshadri constant can be computed exactly.

The authors also think that the introduction of the numbers α_r and β_r, the realization that the problem has an infinite order automorphism when r is even, $r \geqslant 8$, and the graphical reasoning from §2.1–§2.2 greatly simplify the analysis of the problem.

1.10. Relation with other work, II

The paper [5] gives lower bounds on $\epsilon_r(L)$ for those L whose Seshadri constant is not affected by (-1)-curves. When r is odd, this means line bundles L with $\frac{e_2}{e_1} \in [\frac{2}{(\sqrt{r}-1)^2}, \frac{(\sqrt{r}-1)^2}{2}]$. Then [5, Theorem 5] gives the lower bound $\epsilon_r(L) \geqslant \eta_r(L) \cdot (1 - \frac{1}{5r})^{\frac{1}{2}}$. When r is even, the results of [5] apply to all inner bundles, and [5, Theorem 4] gives the lower bound $\epsilon_r(L) \geqslant \eta_r(L) \cdot (1 - \frac{2}{9r})^{\frac{1}{2}}$.

As discussed in §1.8 in this case we are able to find inner bundles L where $\epsilon_r(L) = \eta_r(L)$. Using these bundles and convexity of the Seshadri constant then gives lower bounds on ϵ_r which are better than the lower bound above on various regions of the intervals above. See the discussion in §10.2.

Limitations of this paper. In the study of Seshadri problems on blowups of rational surfaces, and in particular the Nagata conjecture, the sticking point is our inability to either rule out all K_X-positive curves C with $C^2 < 0$, or exhibit one which exists. Unfortunately this paper is no exception.

However, the construction of the inner square-zero nef classes in §9–§10 does eliminate a large range of such classes, and seems to the authors to be a useful step forward.

Second, the most precise results about Seshadri constants in this paper are for outer bundles, those bundles $L = \mathcal{O}_Y(e_1, e_2)$ with $\frac{e_2}{e_1} \notin [\beta_r, \alpha_r]$. As $r \to \infty$ we have $\beta_r \to 0$ and $\alpha_r \to \infty$. Thus, as r increases, the region of our ignorance also increases, and the region of complete understanding shrinks to zero.

1.11. Organization of the paper

In §2 we describe a graphical way of representing and arguing about the problem and give a heuristic argument that irreducible curves affecting the Seshadri constant of outer bundles should be (-1)-curves. This picture also explains the appearance of α_r and β_r in the problem.

In §3 we show that T_r is an automorphism of the problem when r is even, calculate the nef and effective cones for classes whose slope is outside of $[\beta_r, \alpha_r]$, and compute the Seshadri constants for outer bundles for even $r \geqslant 10$. In §4 we compute the Seshadri constants for outer bundles for odd $r \geqslant 9$. In §5 we give the results for all r, even and odd, with $r \leqslant 8$.

In §6 we study the slopes $\frac{q_{n+1,r}}{q_{n,r}}$ which show up in the exceptional case (*ii*) in the symplectic packing problem (§1.3), as well as establish the properties of the map φ_r defined in (1.8).

In §7 we use the results of the previous sections to establish the results on symplectic packings, Theorems 1.2 and 1.3. In §8 we give the reflection theorem, a method of producing nef classes using certain types of specializations. In §9 we use the reflection theorem to construct inner square-zero nef classes in both the odd and even cases. Finally in §10 we use pullback maps to produce other families of inner square-zero nef classes when r is even, and an interesting family of bounds when $8 \mid r$.

2. The square-zero cone and graphical arguments

2.1. The square-zero cone

Let X be the blowup of $\mathbb{P}^1 \times \mathbb{P}^1$ at r general points. As in §1.3, let $V_r \subset H^2(X, \mathbb{R})$ be the real subspace spanned by the pullbacks F_1, F_2, of the fibre classes from $\mathbb{P}^1 \times \mathbb{P}^1$, and the sum E of the exceptional divisors. We are interested in studying the restriction of the nef and effective cones to V_r.

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Let $C = d_1F_1 + d_2F_2 - mE$ be a class in V_r. If $d_1 < 0$ or $d_2 < 0$ then C is neither effective nor nef. If $d_1, d_2 \geqslant 0$, but $m < 0$, then C is effective but not nef. The real interest is therefore when d_1, d_2, and $m \geqslant 0$, and we restrict to that octant from now on.

In that octant a key object of interest for us is the *square-zero cone*, those classes $\xi \in V_r$ such that $\xi^2 = 0$. A picture of this cone in the octant where $d_1, d_2, m \geqslant 0$ is shown in Figure 1 below.

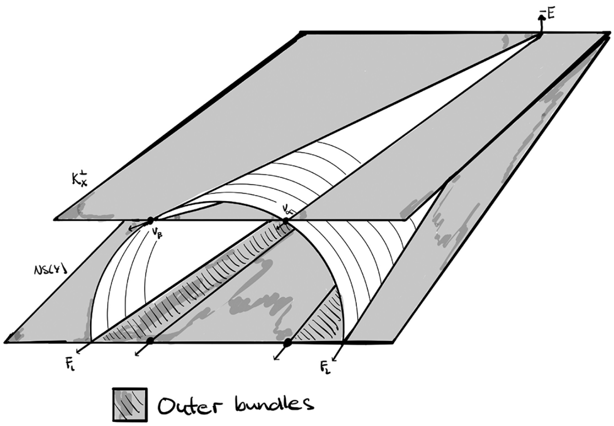

Figure 1. *The square-zero cone, $r \geqslant 9$.*

In the picture, the plane on the base is the subspace spanned by F_1 and F_2, that is, the image of the real Néron-Severi group of $\mathbb{P}^1 \times \mathbb{P}^1$ under the pullback map to X. The curved shape is the square-zero cone, and it meets the base plane in the rays spanned by F_1 and F_2.

The plane at the top of the picture is the subspace of classes orthogonal to K_X, that is, those classes ξ so that $\xi \cdot K_X = 0$. The K_X-negative classes lie below the plane, and the K_X-positive classes lie above.

When $r \leqslant 7$ this plane lies strictly above the square-zero cone, when $r = 8$ this plane is tangent to the cone, and when $r \geqslant 9$ this plane intersects the cone in two rays. These rays are the rays spanned by the vectors v_{α_r} and v_{β_r} defined in (1.6).

Table 1. *Intersections of classes.*

$\cdot$	v_α	v_β	K_X
v_α	0	$\frac{r-8}{r}$	0
v_β	$\frac{r-8}{r}$	0	0
K_X	0	0	$8-r$

The intersection matrix for v_{α_r}, v_{β_r} and K_X ($= v_1$ in the notation in (1.6)) is shown in Table 1 above, and is easily verified from the formulas for those classes, and the formula (1.4) for the intersection form.

This is perhaps the quickest way to check that v_{α_r} and v_{β_r} span the rays above. The table shows that they are both square-zero classes, and orthogonal to K_X. Note that when $r = 8$ we have $v_{\alpha_8} = v_{\beta_8} = -\frac{1}{4}K_X$; this is the case where the plane $K_X^\perp$ is tangent to the square-zero cone.

The projection of the rays spanned by v_α and v_β onto the base plane are rays of slopes α_r and β_r respectively. Those rays in the base plane whose slopes are outside of $[\beta_r, \alpha_r]$ are the outer bundles, and those with slopes in the interval $[\beta_r, \alpha_r]$ are the inner bundles (§1.3).

2.2. Three graphical arguments

There are several places in the paper where an argument can be simply expressed by a picture which would otherwise require a chain of uninformative inequalities. This graphical way of thinking has also guided our approach to the problem. In this subsection we explain our graphical notation, and several elementary facts which can be seen from this point of view.

We restrict ourselves to the octant $d_1, d_2, m \geqslant 0$ of §2.1. The nef and effective cones are stable under scaling by positive real numbers, and so it is sufficient to consider Figure 1 up to scaling, which we represent as a diagram of the type in Figure 2.

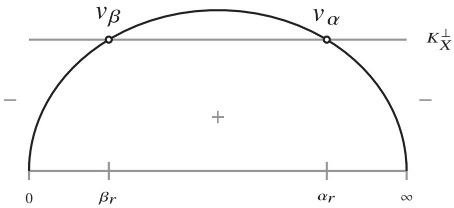

Figure 2. *Figure 1 up to scaling.*

In this picture the curve represents the square-zero cone, the line on the bottom the portion of the nef cone spanned by F_1 and F_2, and the upper line the plane $K_X^\perp$. We label a class $(e_1, e_2, 0)$ along the bottom by its slope $\frac{e_2}{e_1}$, so that F_1 corresponds to slope 0 and F_2 to slope ∞.

The signs + and – in this diagram are a reminder that classes inside the square-zero cone have positive self-intersection, and classes outside have negative self-intersection, and we will omit them from further diagrams. We will also sometimes omit the line for $K_X^\perp$.

Here are three arguments we will use frequently. We first give the associated pictures, and then explain what the statements are.

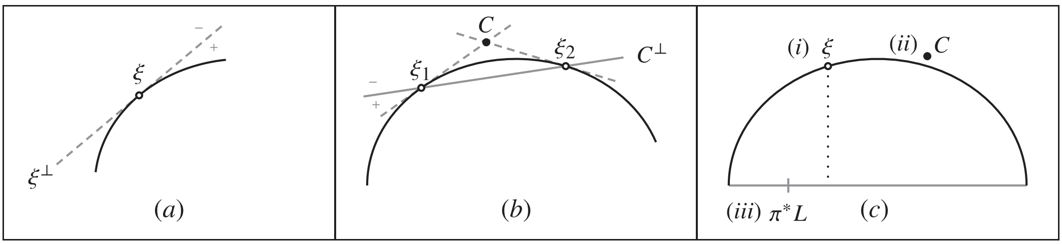

Figure 3. *Three graphical arguments.*

(a) *If ξ is a class on the square-zero cone, then the hyperplane $\xi^\perp$ is the tangent line to the cone at ξ.*

This is the well-known fact that if a variety Q is given as the zeros of a quadratic form $\langle \cdot, \cdot \rangle$ on some vector space, then for any point $x \in Q$, the tangent plane to Q at x consists of those vectors v such that $\langle x, v \rangle = 0$ (since then $\langle x + \epsilon v, x + \epsilon v \rangle$ vanishes to first order in ϵ).

The classes which intersect ξ positively are below this line, and the ones which intersect ξ negatively are above.

(b) *If C is a class with $C^2 < 0$, then the hyperplane $C^\perp$ is spanned by ξ_1 and ξ_2, where ξ_1, ξ_2 are the two points on the square-zero cone whose tangent lines contain C.*

By (a) the intersection of both ξ_1 and ξ_2 with C is zero, therefore all classes on the plane spanned by ξ_1 and ξ_2 intersect C in zero. By reason of dimension, this plane is all of $C^\perp$. The classes above the line, including C itself, intersect C negatively, and the classes below intersect C positively.

(c) *If ξ is a class on the square-zero cone which is nef* (i), *then no effective class C which is to the right of ξ* (ii) *can effect the Seshadri constant of a line bundle L which is to the left of ξ* (iii), *and similarly with right and left reversed in* (ii) *and* (iii).

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The visual interpretation of (1.1) is that one starts at π^*L, and moves upwards in the direction of $-E$ until L_γ $(:= \pi^*L - \gamma E)$ either runs into a plane of the type $C^\perp$ with $C^2 < 0$ or hits the square-zero cone (e.g., see ($c3$) in Figure 4 below). In the first case, $\epsilon_r(L)$ is computed by C (if $C^\perp$ is the first such plane encountered) and in the second case $\epsilon_r(L)$ is the maximum possible value, $\eta_r(L)$.

With reference to Figure 4 below, the argument for (c) is then that, since ξ is nef, the class C must be below $\xi^\perp$ ($c1$). But this means that ξ_1 and ξ_2, the points on the square-zero cone whose tangent lines contain C, must both be to the right of ξ ($c2$). Therefore the line spanned by ξ_1 and ξ_2 exits the square-zero cone to the right of ξ (at worst at ξ if C is on $\xi^\perp$) and so $C^\perp$ (for this C) cannot affect the Seshadri constant of L ($c3$).

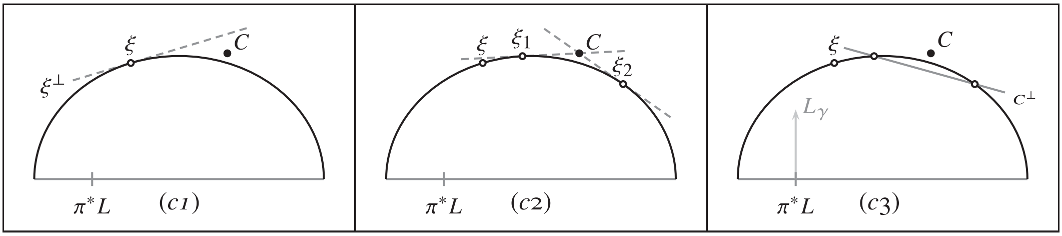

Figure 4. *Argument for Figure 3(c).*

2.3. Reasons for interest in the square-zero cone

The square-zero cone is a natural upper (respectively, lower) bound for the nef cone (respectively the effective cone). The nef cone can extend at most up to the square-zero cone, although it is not clear how close it can get, and the effective cone extends past the square-zero cone, although it is not clear how far.

Second, if ξ is a class on the square-zero cone which is nef, then not only is ξ an example of an extreme nef class (one which reaches the maximum possible boundary), but, by §2.2(c) above, ξ also splits the problem of understanding the nef and effective cones into two pieces. Essentially, there is no information transfer across the dotted line in Figure 3(c); knowledge about nef or effective classes on one side does not allow one to conclude anything about nef or effective classes on the other side. An exception to this principle is when one can use T_r to transport information from one part of the cone to another.

Finally, we note that a square-zero class ξ which is nef is not only an example of an extreme nef class, but it is also an example of an extreme effective class. If ξ is nef, the effective cone must lie below the tangent line at ξ as in Figure 3(a), and so at ξ the effective cone is pinched down to ξ. That is, at such a point the boundaries of the nef and effective cones coincide.

2.4. A heuristic argument

In this section we provide an argument, with several gaps, which suggests the following principle :

If L is an outer bundle on $\mathbb{P}^1 \times \mathbb{P}^1$, and if $\epsilon_r(L) \neq \eta_r(L)$, then the Seshadri constant of L is computed by a (-1) curve (equivalently, by the symmetrization of a (-1)-curve).

The argument is the following. Consider L_γ $(:= \pi^*L - \gamma E)$ for increasing γ. Since L is an outer bundle, L_γ exits the square-zero cone before it crosses the line $K_X^\perp$, as in Figure 5. If $\epsilon_r(L) \neq \eta_r(L)$ then $\epsilon_r(L)$ is computed by some irreducible curve C', with symmetrization C (as in §1.3). By [4, Theorem 2.6.2(f)] C must be quite close to the square-zero cone.

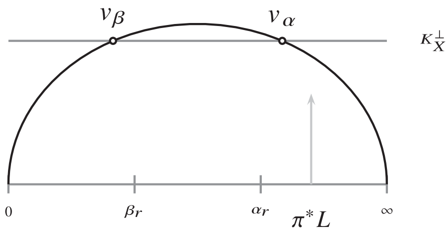

Figure 5. *Situation of the heuristic argument.*

Thus the line $C^\perp$ (as in Figure 3(b)) must be quite small, and so C quite close to the point where L_γ exits the square-zero cone. This suggests that C will also be below the line $K_X^\perp$, and so K_X-negative. If so, then C' is also K_X-negative (all curves in the symmetrization have the same intersection with K_X).

Therefore we have $(C')^2 \leqslant -1$ and $C' \cdot K_X \leqslant -1$. On a smooth irreducible surface X, and with C' an irreducible curve, one always has $(C' + K_X) \cdot C' \geqslant -2$. Thus both inequalities above are equalities, and C' is a (-1)-curve.

Despite the gaps in the argument above, the conclusion is correct. When r is even we show in Corollary 3.4 that both v_{α_r} and v_{β_r} are nef classes. As a result, first, by §2.2(c) only curves C whose slopes are outside $[\beta_r, \alpha_r]$ can affect the Seshadri constant of an outer bundle. Second, since v_{α_r} and v_{β_r} are nef, any symmetric effective curve C must be below the tangent lines to v_{α_r} and v_{β_r} (§2.2(a)), and therefore a curve C with slope outside $[\beta_r, \alpha_r]$ is K_X-negative, as suggested in §2.4.

In the case that r is odd (and $r \neq 9$) the classes v_{α_r} and v_{β_r} are not nef, and we require a different argument to show that K_X-positive curves (or K_X-null curves) cannot influence the Seshadri constant of any outer bundle. This argument appears in §4.2–§4.3.

3. Even *r*, $r \geqslant 10$

3.1. Theorem on automorphisms when r is even

The following result is the key to our analysis of the Seshadri constants of outer bundles when r is even.

Theorem 3.1. *Let $\pi\colon X \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ be the blowup of $\mathbb{P}^1 \times \mathbb{P}^1$ at r general points, $p_1, \ldots, p_r$, with r even. As in §1.3 let $V_r \subset H^2(X, \mathbb{R})$ be the subspace generated by the fibre classes F_1 and F_2, along with the sum of the exceptional divisors E. Then*

(a) *The linear transformation $T_r : V_r \longrightarrow V_r$ given, in the basis F_1, F_2, E by*

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & \frac{r}{2} & r \\ 0 & -1 & -1 \end{bmatrix}$$

is an automorphism of V_r, preserving the intersection form.

(b) *The eigenvalues of T_r are α_r, β_r, and 1, with respective eigenvectors v_{α_r}, v_{β_r}, and K_X, where v_{α_r} and v_{β_r} are the vectors given in (1.6).*

(c) *If $\xi \in V_r$ is any class, then ξ is nef, or effective, or represents a curve with s irreducible components if and only if $T_r(\xi)$ is respectively nef, effective, or represents a curve with s irreducible components.*

Thus, by (c), when r is even T_r induces an automorphism of the nef and effective cones restricted to V_r. This automorphism is of infinite order whenever $r \geqslant 8$.

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Proof. (a) The identity

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & \frac{r}{2} & r \\ 0 & -1 & -1 \end{bmatrix}^t \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -r \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & \frac{r}{2} & r \\ 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -r \end{bmatrix},$$

where t denotes matrix transpose, shows that T_r preserves the intersection form on V_r.

(b) The characteristic polynomial of T_r is $\big(t^2 - (r-4/2)t + 1\big)(t-1)$, where now t denotes a variable, and therefore the eigenvalues of T_r are α_r, β_r, and 1. It is straightforward to verify that K_X is an eigenvector of eigenvalue 1. Using the identity $\alpha_r^2 = (r-4/2)\alpha_r - 1$ as is, and in the form $2(\alpha_r+1) - r\alpha_r = -2(\alpha_r+1)\alpha_r$, we compute that

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & \frac{r}{2} & r \\ 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{\alpha_r+1} \\ \frac{\alpha_r}{\alpha_r+1} \\ -\frac{2}{r} \end{bmatrix} = \begin{bmatrix} \frac{\alpha_r}{\alpha_r+1} \\ \frac{\left(\frac{r-4}{2}\right)\alpha_r - 1}{\alpha_r+1} \\ \frac{2}{r} - \frac{\alpha_r}{\alpha_r+1} \end{bmatrix} = \begin{bmatrix} \frac{\alpha_r}{\alpha_r+1} \\ \frac{\alpha_r^2}{\alpha_r+1} \\ \frac{-2(\alpha_r+1)\alpha_r}{r(\alpha_r+1)} \end{bmatrix} = \alpha_r \begin{bmatrix} \frac{1}{\alpha_r+1} \\ \frac{\alpha_r}{\alpha_r+1} \\ -\frac{2}{r} \end{bmatrix},$$

and so $T_r(v_{\alpha_r}) = \alpha_r\, v_{\alpha_r}$. Similarly $T_r(v_{\beta_r}) = \beta_r\, v_{\beta_r}$.

The real value of the theorem is in part (c). The idea of the argument is that the proper transform of each fibre of type F_1 passing through a point p_i is a (-1)-curve. Blowing down these r-different (-1)-curves gives another way to realize X as a blowup of $\mathbb{P}^1 \times \mathbb{P}^1$. Comparing the two descriptions as blowups and switching the factors of $\mathbb{P}^1 \times \mathbb{P}^1$ gives T_r.

To carry this out, consider the linear series $|\mathcal{O}_Y(\frac{r}{2}, 1)|$ on $Y = \mathbb{P}^1 \times \mathbb{P}^1$, of dimension $r+1$. The curves in the series have self-intersection r, and intersection number 1 with curves in $|\mathcal{O}_Y(1,0)|$. When the r points are general, the series $|\pi^*\mathcal{O}_Y(\frac{r}{2}, 1) - E|$ (i.e., the proper transforms of the curves in the series passing through the points) is therefore a basepoint free pencil of curves on X. The curves have self-intersection 0, and intersection number 1 with F_1.

The pencils $|F_1|$ and $|\pi^*(\mathcal{O}_Y(\frac{r}{2}, 1) - E|$ give a birational morphism $\mu\colon X \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1$. This map blows down the curves of class $F_1 - E_i$, $i = 1, \ldots, r$, since these classes have intersection number 0 with the curves in each pencil. Since μ is birational, and since the Picard ranks of X and $\mathbb{P}^1 \times \mathbb{P}^1$ are $r+2$ and 2 respectively, these are the only curves blown down.

Thus μ also expresses X as the blowup of $\mathbb{P}^1 \times \mathbb{P}^1$ at r points, say at $q_1, \ldots, q_r$. (The map μ is only really well-defined when we have fixed bases for these pencils; we will do this below.) Let F_1', F_2' and E' be the pullback of the fibre classes via μ, and the sum of the exceptional divisors of μ respectively. The matrix expressing the change of coordinates on V_r from the second basis to the first is

$$T_r' := \begin{array}{c} \\ F_1 \\ F_2 \\ E \end{array} \!\!\begin{array}{c} \begin{array}{ccc} F_1' & F_2' & E' \end{array} \\ \begin{bmatrix} 1 & \frac{r}{2} & r \\ 0 & 1 & 0 \\ 0 & -1 & -1 \end{bmatrix} \end{array}.$$

Since this matrix represents the identity transformation on V_r, albeit between two different bases, a vector v (in the basis F_1', F_2', and E') is nef, or effective, or represents a curve with s irreducible components if and only if the vector $T_r'(v)$ (in the basis F_1, F_2, and E) respectively is nef, or effective, or represents a curve with s irreducible components.

If $p_1, \ldots, p_r$ are in very general position, then $q_1, \ldots, q_r$ are also in very general position. We will check this below, but first show how this is enough to finish the argument.

If $p_1, \ldots, p_r$, and $q_1, \ldots, q_r$ are in very general position, then a class $d_1F_1 + d_2F_2 - mE$ is nef, effective, or represents a curve with s irreducible components if and only if the class $d_1F_1' + d_2F_2' - mE'$ has the respective property.

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Thus the matrix

$$T_r'' := \begin{array}{c} \\ F_1 \\ F_2 \\ E \end{array}\!\!\begin{array}{c} \begin{array}{ccc} F_1 & F_2 & E \end{array} \\ \begin{bmatrix} 1 & \frac{r}{2} & r \\ 0 & 1 & 0 \\ 0 & -1 & -1 \end{bmatrix} \end{array} = \begin{array}{c} \\ F_1 \\ F_2 \\ E \end{array}\!\!\begin{array}{c} \begin{array}{ccc} F_1' & F_2' & E' \end{array} \\ \begin{bmatrix} 1 & \frac{r}{2} & r \\ 0 & 1 & 0 \\ 0 & -1 & -1 \end{bmatrix} \end{array} \cdot \begin{array}{c} \\ F_1' \\ F_2' \\ E' \end{array}\!\!\begin{array}{c} \begin{array}{ccc} F_1 & F_2 & E \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

gives a linear transformation $V_r \longrightarrow V_r$ in the basis (F_1, F_2, E) preserving each of those properties.

The transformation T_r'' is a reflection. The transformation S_r which fixes E and switches F_1 and F_2 (i.e., the transformation induced by the automorphism of $\mathbb{P}^1 \times \mathbb{P}^1$ switching the factors) is also a reflection, and also preserves all the properties we are interested in. The product $S_r \cdot T_r''$ is T_r, and therefore T_r preserves classes which are nef, or effective, or which represent curves with s irreducible components, as claimed.

Thus, to complete the proof of (c) it is sufficient to verify that $q_1, \dots, q_r$ are in very general position if $p_1, \dots, p_r$ are. This is clear when $r = 2$ (any two points not on the same fibres are in very general position), and so from now on we assume that $r \geqslant 4$.

By acting by $\mathrm{Aut}(\mathbb{P}^1) \times \mathrm{Aut}(\mathbb{P}^1)$ we may assume that $p_1 = ([1:0],[1:0])$, $p_2 = ([0:1],[0:1])$, and $p_3 = ([1:1],[1:1])$. Similarly we may choose a basis for the pencils $|F_1|$ and $|\pi^*(\mathcal{O}_Y(\frac{r}{2},1) - E|$ so that under the map μ, $q_1 = ([1:0],[1:0])$, $q_2 = ([0:1],[0:1])$, and $q_3 = ([1:1],[1:1])$, where q_i is the image of $F_1 - E_i$. This choice is enough to fix the map μ uniquely.

Let $U \subset (\mathbb{P}^1 \times \mathbb{P}^1)^{r-3}$ be the Zariski open subset set of the configuration space of $r-3$ points $(p_4, \dots, p_r)$ so that $p_1, \dots, p_r$ (with p_1, p_2, and p_3 as above) are distinct, and such that the curves in $|\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(\frac{r}{2},1)|$ passing through $p_1, \dots, p_r$ form a pencil whose generic member is irreducible.

By T_r' above, $\frac{r}{2}F_1' + F_2' - E' = F_2$, and thus the points $q_1, \dots, q_r$ (with q_1, q_2, q_3 also as above) are sufficiently independent so that the linear series $|\frac{r}{2}F_1' + F_2' - E'|$ is a basepoint-free pencil of curves, generically irreducible.

Thus the process $(p_1, \dots, p_r) \mapsto (q_1, \dots, q_r)$ induces a map

$$I \colon (\mathbb{P}^1 \times \mathbb{P}^1)^{r-3} \longrightarrow (\mathbb{P}^1 \times \mathbb{P}^1)^{r-3}$$

such that $I(U) \subseteq U$. Moreover, the identity $\frac{r}{2}F_1' + F_2' - E' = F_2$ shows that $I(q_4, \dots, q_r) = (p_4, \dots, p_r)$, that is. that I is an involution.

Therefore given a family $V_n \subset U$, $n \in \mathbb{N}$, of proper closed subsets of U, the set

$$W := \bigcup_{n\in\mathbb{N}} (V_n \cup I(V_n))$$

is a countable union of proper closed subsets of U, stable under I. Thus if $(p_4, \dots, p_r) \notin W$, then $(q_4, \dots, q_r) = I(p_4, \dots, p_r) \notin W$. Therefore, if $p_1, \dots, p_r$ are very general, then so are $q_1, \dots, q_r$. This finishes the proof of (c).

When $r = 8$, T_8 is a unipotent matrix consisting of a single Jordan block. When $r \geqslant 10$, α_r is a real number with $\alpha_r > 1$. Thus T_r is of infinite order for all $r \geqslant 8$. □

Remarks 3.2. (1) The process of blowing down, and switching factors in the proof of Theorem 3.1(c) gives an automorphism $\tilde{T}_r$ of all of $H^2(X, \mathbb{R})$ (or $H^2(X, \mathbb{Z})$), and not just V_r. In $H^2(X, \mathbb{R})$ the orthogonal complement to V_r is the subspace $0 \cdot F_1 + 0 \cdot F_2 - \sum_{i=1}^r m_i E_i$ with $\sum_i m_i = 0$. On this subspace one can check that $\tilde{T}_r$ acts as multiplication by -1.

(2) When $r \geqslant 9$, $\mathbb{P}^2$ blown up at r general points (and $\mathbb{P}^1 \times \mathbb{P}^1$ blown up at an odd number of points) has many such infinite order Cremona "relabelling" automorphisms acting on H^2 of the blowup. The great advantage in the case of $\mathbb{P}^1 \times \mathbb{P}^1$ blown up at an even number of points is that $\tilde{T}_r$ preserves the subspace of equal multiplicity curves. No such automorphisms exist in the cases of $\mathbb{P}^2$ blown up at r general points, nor for $\mathbb{P}^1 \times \mathbb{P}^1$ blown up at an odd number of points.

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In the case of $\mathbb{P}^2$, the corresponding space V_r is two dimensional, spanned by the hyperplane class H and E. Since H is on the boundary of the nef cone, any such automorphism has to take H to H. But to preserve the intersection form, the automorphism must now take E to $\pm E$, and in order to preserve the nef cone, it must take E to $+E$. Thus, on $\mathbb{P}^2$ blown up at any number of points, any such automorphism which preserves the equal multiplicity subspace acts as the identity on V_r. We will see in §4.3 that, other than switching the factors, there is no such automorphism for $\mathbb{P}^1 \times \mathbb{P}^1$ blown up at an odd number of points.

(3) Parts (a) and (b) of Theorem 3.1 still hold when r is odd. However, since T_r does not preserve nef or effective classes when r is odd, this transformation is meaningless for our problem.

Theorem 3.3 (nef cone for outer bundles, r even). *Let X be the blowup of $\mathbb{P}^1 \times \mathbb{P}^1$ at r general points, with $r \geqslant 10$ even. Then the nef cone in V_r, restricted to the half plane $K_X^{\leqslant 0}$ is spanned by v_{α_r}, v_{β_r}, and the classes $T_r^n(F_2)$, with $n \in \mathbb{Z}$.*

Proof. For $n \in \mathbb{Z}$ set $\xi_n := T_r^n(F_2)$. Since F_2 is a nef, square-zero class, intersecting K_X negatively, by Theorem 3.1 each of the ξ_n is also a nef square-zero class intersecting K_X negatively.

Since $F_2 \cdot v_{\beta_r} = \frac{\alpha_r}{\alpha_r+1}$, we conclude from Table 1 that when writing $F_2 \in V_r$ in the basis v_{α_r}, v_{β_r}, and K_X, the coefficient of v_{α_r} is $\frac{r}{r-8} \cdot \frac{\alpha_r}{\alpha_r+1}$. Thus, since α_r is the dominant eigenvalue of T_r, we have

$$v_{\alpha_r} = \frac{(r-8)(\alpha_r+1)}{r \cdot \alpha_r} \lim_{n\to\infty} \frac{1}{\alpha_r^n} \xi_n,$$

and so v_{α_r} is nef. Similarly, v_{β_r} is nef. We have already checked that both classes are square-zero.

Therefore the intersection of the nef cone and $K_X^{\leqslant 0}$ contains the convex hull of v_{α_r}, v_{β_r}, and the classes ξ_n, $n \in \mathbb{Z}$, as in Figure 6(a) below.

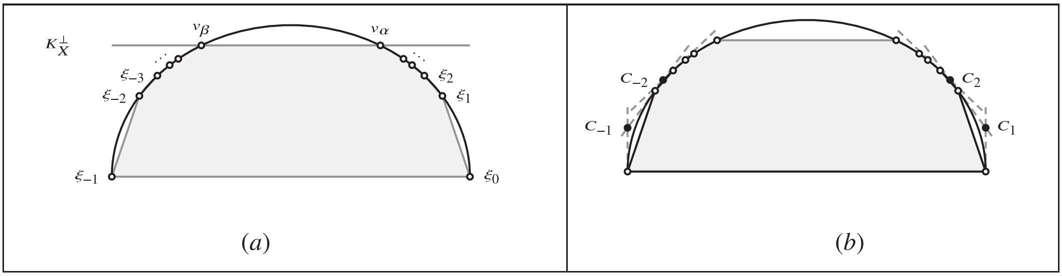

Figure 6. *Argument for Theorem 3.3.*

Next, for $n \in \mathbb{Z}$ set $C_n := T_r^n(E)$. Since $F_1 \cdot E = 0$ and $F_2 \cdot E = 0$, that is, since $\xi_{-1} \cdot C_0 = 0$ and $\xi_0 \cdot C_0 = 0$, and since T_r preserves intersections, we have $\xi_{n-1} \cdot C_n = 0$ and $\xi_n \cdot C_n = 0$ for all $n \in \mathbb{Z}$.

By 2.2(b) this means that for each n, the nef cone cannot pass the line spanned by ξ_{n-1} and ξ_n. Thus (as illustrated in Figure 6(b)) the intersection of the nef cone with $K_X^{\leqslant 0}$ can be no larger than the cone spanned by v_{α_r}, v_{β_r}, and the ξ_n, $n \in \mathbb{Z}$. □

Corollary 3.4 (Case of even r in Theorem 1.1). *Let X be the blowup of $\mathbb{P}^1 \times \mathbb{P}^1$ at r general points, with $r \geqslant 10$ even.*

(a) *The classes v_{α_r} and v_{β_r} are nef.*
(b) *For any ample bundle $L = \mathcal{O}_Y(e_1, e_2)$ with $\frac{e_2}{e_1} \notin [\beta_r, \alpha_r]$, the Seshadri constant of L is computed by C_n, where n is an integer such that $L_{\eta_r(L)}$ is between ξ_n and ξ_{n-1} on the square zero cone.*

Proof. Part (a) was established in the proof of Theorem 3.3, and is included here for reference. Part (b) is immediate from the visual interpretation of (1.1) (as in §2.2(c)), and the description of the nef cone in Theorem 3.3.

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In more detail, the line segment L_γ, $\gamma \geqslant 0$, meets the square-zero cone when $\gamma = \eta_r(L)$. By Theorem 3.3 if L is an outer bundle the line L_γ exits the nef cone through a secant line spanned by ξ_n and ξ_{n-1} for some n (unique unless $L_{\eta_r(L)}$ is one of the ξ_m), as illustrated in Figure 7. Thus $L_{\eta_r(L)}$ is between ξ_n and ξ_{n-1}, and can be used to identify n.

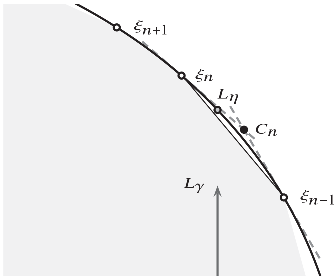

Figure 7. *Diagram for Corollary 3.4 (b).*

The proof of Theorem 3.3 shows that this secant line is also the line $C_n^\perp$, and therefore C_n computes the Seshadri constant of L. □

3.2. Calculating the Seshadri constant of an outer bundle, r even

Given a line bundle $L = \mathcal{O}_Y(e_1, e_2)$ in order to use Corollary 3.4(b) to compute $\epsilon_r(L)$ one needs to find the correct value of n, and then find C_n.

Since $C_n = T_r^n(E)$, the coordinates of the C_n satisfy the recursion relation coming from the characteristic polynomial of T_r, that is, satisfy (1.9). It follows that, as in §1.5, if one defines sequences $\{p_n\}_{n\in\mathbb{Z}}$ and $\{m_n\}_{n\in\mathbb{Z}}$ by $p_{-1} = 0$, $p_0 = 0$, $p_1 = r$, and $m_{-1} = 1$, $m_0 = -1$, $m_1 = 1$, and the recursion (1.9), that for all n the class of C_n is $(p_{n-1}, p_n, -m_n)$. If $L = \mathcal{O}_Y(e_1, e_2)$, $\epsilon_r(L)$ is then computed by (1.11), for the right value of n.

One can use several methods to find n. If one considers the square zero cone in the region where d_1, $d_2 \geqslant 0$ (including $m < 0$, that is, not just in the octant $m \geqslant 0$ from §2.1) and removes the rays spanned by v_{α_r} and v_{β_r}, then up to scaling by $\mathbb{R}_{>0}$ what remains is the union of two disjoint open intervals. The map φ_r defined by (1.7) gives a diffeomorphism of each of these intervals with $\mathbb{R}$, converting the action of T_r into addition by 1, and converting the operation of swapping the fibre classes into multiplication by -1 (see §6.3).

It is straightforward to check that $\varphi_r(F_2) = \frac{1}{2}$, from which it follows that $\varphi_r(\xi_n) = n + \frac{1}{2}$ for all $n \in \mathbb{Z}$. Thus, if $L_{\eta_r(L)}$ is between ξ_n and ξ_{n-1} on the square zero cone, we have $n + \frac{1}{2} \geqslant \varphi_r(L_{\eta_r(L)}) \geqslant n - \frac{1}{2}$ and so (as long as $L_{\eta_r(L)} \neq \xi_n$), $n = \lfloor \varphi_r(L_{\eta_r(L)}) + \frac{1}{2} \rfloor$. If $L_{\eta_r(L)} = \xi_n$, then this formula produces $n+1$ instead of n, but in this case $\epsilon_r(L) = \eta_r(L)$, and both C_n and C_{n+1} compute this answer. This is the method given in Theorem 1.1.

The previous method has the advantage of giving a formula for n; however, it is not very useful computationally. Evaluating φ_r correctly requires a high degree of precision in real number calculations, too large to be of much use in general. Instead, it is computationally more efficient to find n so that the slope of L is between the slopes of ξ_n and ξ_{n-1}.

As in §1.7, define a sequence $\{q_n\}_{n\in\mathbb{Z}}$ by $q_{-1} = 1$, $q_0 = 0$, $q_1 = 1$, and the recursion (1.9). Then $\xi_n = T_r^n(F_2) = (q_n, q_{n+1}, -\sqrt{\frac{2q_{n+1}q_n}{r}})$ for all $n \in \mathbb{Z}$, and thus the slope of ξ_n is $\frac{q_{n+1}}{q_n}$ (this is the reason for case (ii) in Theorem 1.3).

If $L = \mathcal{O}_Y(e_1, e_2)$, with $e_1 \leqslant e_2$ and $e_1 \neq 0$, then the relevant value of n is the smallest $n \geqslant 1$ so that $\frac{q_{n+1}}{q_n} \leqslant \frac{e_2}{e_1}$, that is, the smallest $n \geqslant 1$ so that $e_2 q_n - e_1 q_{n+1} \geqslant 0$. Computing the q_n's and checking the previous condition only involves integer arithmetic.

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If instead $e_1 > e_2$, one can either use the fact that, by symmetry, $\epsilon_r(L) = \epsilon_r(\mathcal{O}_Y(e_2, e_1))$ and the method above, or use a similar argument for negative n.

3.3. Automorphisms preserving V_r

In light of the utility of T_r it is interesting to ask if there are other integral linear automorphisms of V_r which preserve all aspects of the problem (i.e., which preserve the intersection form, the nef and effective cones, and the canonical class). Let G denote the group of such automorphisms.

In the proof of Theorem 3.1(c) we have seen that S_r, the automorphism switching F_1 and F_2 and fixing E, is in G. The following argument shows that T_r and S_r generate G. Since $S_r T_r S_r^{-1} = T_r^{-1}$ (and $S_r = S_r^{-1}$), effectively this means that, up to switching F_1 and F_2, there are really no linear automorphisms of the problem other than the T_r^n.

Proof of claim. Suppose that $g \in G$, and consider $g(\xi_0)$. Since g preserves the intersection form, the canonical class, and the nef cone, $g(\xi_0)$ is a square-zero nef class which intersects K_X negatively. By Theorem 3.3 this means that $g(\xi_0)$ must be a multiple of one of the ξ_n. Since g is an integral linear transformation (i.e., coming from an action on $V_{r,\mathbb{Z}}$, the underlying integral lattice), and since each ξ_n is the integral generator on the ray it spans, we conclude that $g(\xi_0) = \xi_n$ for some unique $n \in \mathbb{Z}$. Multiplying on the left by T_r^{-n} we may assume that $g(\xi_0) = \xi_0$.

Now consider $g(\xi_{-1})$ and $g(\xi_1)$. By the previous reasoning, we must have $g(\xi_{-1}) = \xi_i$ and $g(\xi_1) = \xi_j$ for unique $i, j \in \mathbb{Z}$. Since g preserves the intersection form, we have $\xi_0 \cdot \xi_i = g(\xi_0) \cdot g(\xi_{-1}) = \xi_0 \cdot \xi_{-1} = 1$ and similarly $\xi_0 \cdot \xi_j = 1$.

But, it is easy to verify that the only m for which $\xi_0 \cdot \xi_m = 1$ are $m = -1, 1$. Therefore either $g(\xi_{-1}) = \xi_{-1}$ and $g(\xi_1) = \xi_1$, or $g(\xi_{-1}) = \xi_1$ and $g(\xi_1) = \xi_{-1}$.

In the first case, g now fixes ξ_{-1}, ξ_0, and ξ_1, and so must act as the identity on V_r, since these three classes span V_r. In the second case, $T_r S_r$ is also a transformation which fixes ξ_0 and swaps ξ_{-1} and ξ_1. Multiplying g on the left by $T_r S_r$ then reduces us to the first case. Thus, in both cases g is in the group generated by T_r and S_r. □

4. Odd r, $r \geqslant 9$

4.1. Portrait of the outer nef cone, r odd

Figure 8(a) below shows, for odd r, $r \geqslant 11$, the (-1)-curves, or symmetrizations of (-1)-curves, which can affect the Seshadri constants of line bundles. In contrast to the case when r is even, there are only four such curves; these are labelled $C_1, \ldots, C_4$ below. The picture is somewhat cluttered, so in (b) and (c) we show separately the two curves C_4 and C_3 on the right hand side of (a).

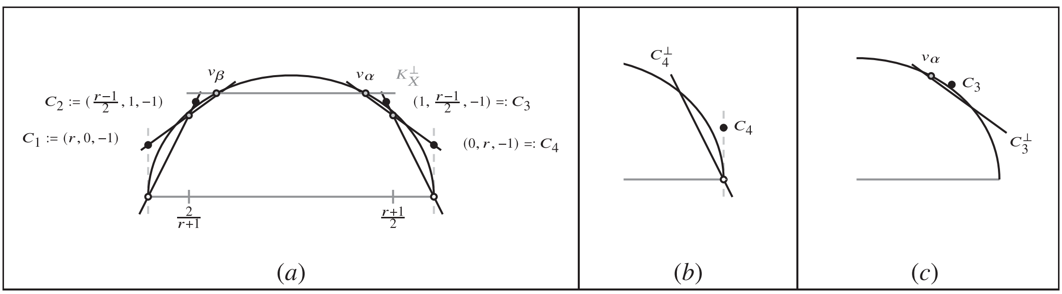

Figure 8. *Curves affecting outer bundles, odd $r \geqslant 9$.*

The curve C_4 of class $(0, r, -1)$ is the union of the proper transforms of the fibres of type F_2 through each of the p_i, that is, it is the union of the r (-1)-curves $F_2 - E_i$, $i = 1, \ldots, r$. The plane $C_4^\perp$ intersects

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the square-zero cone in rays spanned by the classes F_2 and $(2, r, -2)$. A picture of C_4, and $C_4^\perp$ appears in Figure 8(b).

The linear series $|\mathcal{O}_Y(1, \frac{r-1}{2})|$ has dimension r, curves in the series have self-intersection $r-1$, and smooth curves in the series are rational. When the r points are general, $|\pi^*\mathcal{O}_Y(1, \frac{r-1}{2}) - E|$ therefore consists of a single smooth rational curve of self-intersection -1; that is, the class $C_3 = (1, \frac{r-1}{2}, -1)$ is represented by an irreducible (-1)-curve.

The plane $C_3^\perp$ intersects the square-zero cone in rays spanned by the classes

$$\left(1, \frac{(\sqrt{r}-1)^2}{2}, -(1-\frac{1}{\sqrt{r}})\right) \text{ and } \left(1, \frac{(\sqrt{r}+1)^2}{2}, -(1+\frac{1}{\sqrt{r}})\right). \tag{4.1}$$

When r is odd, $r \geqslant 11$, the class v_{α_r} is strictly above the plane $C_3^\perp$, and thus (in contrast to the case when r is even) in those cases neither v_{α_r} nor v_{β_r} are nef. The planes $C_3^\perp$ and $C_4^\perp$ intersect in the ray spanned by the class $(1, \frac{r+1}{2}, -1)$, of slope $\frac{r+1}{2}$. The existence of the effective classes C_3 and C_4 gives the following inequalities on Seshadri constants for a line bundle $L = \mathcal{O}_Y(e_1, e_2)$.

- If $\frac{e_2}{e_1} \in [\frac{(\sqrt{r}-1)^2}{2}, \frac{r+1}{2}]$ then $\epsilon_r(L) \leqslant \frac{(r-1)e_1+2e_2}{2r}$ (inequality imposed by C_3);
- If $\frac{r+1}{2} \leqslant \frac{e_2}{e_1}$ then $\epsilon_r(L) \leqslant e_1$ (inequality imposed by C_4).

Our goal in this section is to show that the above inequalities, and the symmetric inequalities involving C_1 and C_2, are equalities whenever $\frac{e_2}{e_1} \notin (\beta_r, \alpha_r)$. That is, our goal is to prove the following result.

Theorem 4.1 (Case of odd r in Theorem 1.1). *Suppose that r is odd, $r \geqslant 9$, that $L = \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(e_1, e_2)$ with $e_1, e_2 \geqslant 1$, and that L is an outer bundle, that is, that $\frac{e_2}{e_1} \notin (\beta_r, \alpha_r)$. Then*

$$\epsilon_r(L) = \begin{cases} e_2 & \text{if } \frac{e_2}{e_1} \leqslant \frac{2}{r+1}, \\ \frac{2e_1+(r-1)e_2}{2r} & \text{if } \frac{e_2}{e_1} \in [\frac{2}{r+1}, \beta_r], \\ \frac{(r-1)e_1+2e_2}{2r} & \text{if } \frac{e_2}{e_1} \in [\alpha_r, \frac{r+1}{2}], \\ e_1 & \text{if } \frac{r+1}{2} \leqslant \frac{e_2}{e_1}. \end{cases} \tag{4.2}$$

4.2. *Outline of the argument*

The essential claim of the theorem is that there is no effective curve C which, for an outer bundle L, imposes a stronger condition on the Seshadri constant of L than those imposed by $C_1, \ldots, C_4$.

When $r = 9$ the picture is slightly different from that of the general case shown in Figure 8(a). The difference is that v_{α_9} (respectively v_{β_9}) is on the line $C_3^\perp$ (respectively $C_2^\perp$). Specifically,

$$v_{\alpha_9} = \frac{1}{3}\left(1, \frac{(\sqrt{9}-1)^2}{2}, -(1-\frac{1}{\sqrt{9}})\right),$$

(i.e., one-third of the first class in (4.1)) and similarly for v_{β_9}. In the case $r = 9$ we show in Corollary 4.4 that v_{α_9} and v_{β_9} are nef. Thus, by the principle of 2.2(c), no curve C with slope in (β_9, α_9) can affect the Seshadri constant of an outer bundle. This establishes the theorem for $r = 9$.

For $r \geqslant 11$ we may make several reductions. First, by symmetry it is enough to restrict to the case that $e_1 \leqslant e_2$. Second, the effective classes C which could effect a Seshadri constant must satisfy $C^2 < 0$ and be K_X-positive (any irreducible class, or symmetrization of an irreducible class as in §1.3, which is not $C_1, \ldots, C_4$ must be below the planes $C_1^\perp, \ldots, C_4^\perp$, and to also satisfy $C^2 < 0$ must therefore be above the plane $K_X^\perp$).

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We are not able to show that such curves C don't exist. However, if one does exist, we are able to show that for any nef bundle $L = \mathcal{O}_Y(e_1, e_2)$ with $\frac{e_2}{e_1} \geqslant \alpha_r$, that

$$\frac{C \cdot \pi^* L}{C \cdot E} \geqslant \frac{(r-1)e_1 + 2e_2}{2r} = \frac{C_3 \cdot \pi^* L}{C_3 \cdot E}.$$

That is, we are able to show that there are no such curves C which impose a stronger condition on a bundle of slope $\geqslant \alpha_r$ than that imposed by C_3, and this is enough to establish the theorem.

The nonexistence of such a C is the result of an estimate on how strong a condition such a C could put on a bundle of slope α_r, combined with an estimate on the size of α_r. This preliminary material appears below. The concluding arguments of the proof, using these steps, appears in §4.3.

Proposition 4.2 (Weak lower bound on multiplicity for K_X-positive curves). *Let X be the blowup of $\mathbb{P}^1 \times \mathbb{P}^1$ at r general points (r may be even or odd, and there is no restriction on the size of r). Then there does not exist an effective curve C such that $C^2 < 0$, $K_X \cdot C \geqslant 0$ and all multiplicities of C are in $\{0, 1\}$ (i.e., such that the class of C is $d_1F_1 + d_2F_2 - \sum_{i=1}^r m_iE_i$ with each $m_i \in \{0, 1\}$).*

Proof. The vector space of curves of bidegree (d_1, d_2) on $\mathbb{P}^1 \times \mathbb{P}^1$ with multiplicity $m_1, \ldots, m_r$ at p_1, $\ldots$, p_r has expected dimension max of 0 and $(d_1+1)(d_2+1) - \frac{1}{2}\sum_{i=1}^r m_i^2 - m_i$. Letting C be of class $d_1F_1 + d_2F_2 - \sum_{i=1}^r m_iE_i$, this is the same as

$$\max\Big(0, \tfrac{1}{2}(C^2 - K_X \cdot C) + 1\Big). \tag{4.3}$$

General points of multiplicity 1 impose independent conditions. Therefore, if all the multiplicities are 0 or 1, and if the p_i are general, the dimension of the vector space of such curves is the expected one, as given by (4.3). If $C^2 < 0$ and $K_X \cdot C \geqslant 0$ then (4.3) gives zero (if $C^2 = -1$ then $K_X \cdot C > 0$, since $C^2 - K_X \cdot C$ must be even), and therefore no such effective curves exist. □

Lemma 4.3. *Let $\pi\colon X \longrightarrow Y$ be the blowup of a smooth surface Y at r general points, L an integral nef line bundle on Y such that $\eta_r(L) \in \mathbb{Q} \cap [0, 2]$, and set $\tilde{L} = L_{\eta_r(L)} = \pi^*L - \eta_r(L)E$. If there is no irreducible curve C on X with multiplicities in $\{0, 1\}$ such that $\tilde{L} \cdot C < 0$, then $\tilde{L}$ is a square-zero nef class on X.*

Proof. With $[0, 2]$ replaced by $[0, 1]$, this result was previously known, and in that version one does not even need to check for possible C's such that $\tilde{L} \cdot C < 0$. The version above, increasing $[0, 1]$ to $[0, 2]$, but requiring one to eliminate certain possible classes of C, appears as [4, Corollary 2.7.2]. □

Corollary 4.4 (v_{α_9} and v_{β_9} are nef). *Let X be the blowup of $\mathbb{P}^1 \times \mathbb{P}^1$ at 9 general points. Then the classes v_{α_9} and v_{β_9} are nef on X.*

Proof. Setting $L = \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(3, 6)$, we have $\eta_9(L) = 2$ and $\tilde{L} = \pi^*L - 2E = 3F_1 + 6F_2 - 2E = 9v_{\alpha_9}$. By Lemma 4.3 the only way that $\tilde{L}$ could not be nef is if there is an irreducible curve C with multiplicities in $\{0, 1\}$ such that $\tilde{L} \cdot C < 0$. Thus, it suffices to show that there is no such curve where $v_{\alpha_9} \cdot C < 0$; such a curve must satisfy $C^2 < 0$.

By Proposition 4.2 there is no such effective class with $K_X \cdot C \geqslant 0$, and therefore we must have $K_X \cdot C < 0$. But, such curves (with all multiplicities in $\{0, 1\}$ and $C^2 < 0$) are, in the notation of §4.1, either C_2, C_3, or the components which make up C_1 and C_4.

Since $C_3 \cdot v_{\alpha_9} = 0$, and $C_i \cdot v_{\alpha_9} > 0$ for $i = 1, 2, 4$, we conclude that v_{α_9} is nef. Similarly, v_{β_9} is nef. □

Theorem 4.5. *Let $\pi\colon X \longrightarrow Y$ be the blowup of a smooth surface Y at r general points. Let C be the class of an effective curve on X with all multiplicities equal to m, $m \geqslant 2$, and such that C is not the symmetrization (as in §1.3) of a curve C' with all multiplicities in $\{0, 1\}$, and put $t = \frac{m-1}{m^2}$. Then for all nef classes L on Y,*

$$\frac{C\cdot\pi^*L}{C\cdot E}\geqslant\eta_r(L)\cdot\sqrt{\frac{r-t}{r}}. \tag{4.4}$$

Proof. This lower bound, in contrapositive form, is [4, Theorem 4.1.1]. □

For use below we note that this result applies to real nef classes L, and not just integral ones. (For instance, accepting that (4.4) holds for integral nef bundles on Y, since both sides scale the same way, the inequality must also hold for rational nef classes, and then, by continuity, for real nef classes.)

We will need the following elementary estimate, whose proof is left to the reader.

Lemma 4.6. *For $r\geqslant 9$ the estimates*

$$\tfrac{r-5}{2}\leqslant\alpha_r<\tfrac{r-4}{2}$$

hold. Furthermore, the lower bound is strict whenever $r>9$.

Proposition 4.7. *Suppose that $r\geqslant 11$ and that t, with $t\leqslant r$, is such that*

$$\frac{2\alpha_r+(r-1)}{4(\alpha_r+1)}\geqslant\sqrt{\frac{r-t}{r}}.$$

Then $t>\frac{1}{4}$.

Proof. Squaring both sides, and using the identity $\alpha_r^2=(r-4/2)\alpha_r-1$, in that form, and in the form $(\alpha_r+1)^2=\frac{r}{2}\alpha_r$, the inequality above becomes

$$\frac{6(r-2)\alpha_r+(r-1)^2-4}{8r\alpha_r}\geqslant\frac{r-t}{r}$$

or

$$t\geqslant r-\frac{6(r-2)\alpha_r+(r-1)^2-4}{8\alpha_r}=\frac{2(r+6)\alpha_r-(r-1)^2+4}{8\alpha_r}. \tag{4.5}$$

Using Lemma 4.6 gives

$$t-\frac{1}{4}\geqslant\frac{2(r+6)\left(\frac{r-5}{2}\right)-(r-1)^2+4}{8\left(\frac{r-4}{2}\right)}-\frac{1}{4}=\frac{2r-23}{4r-16}. \tag{4.6}$$

When $r\geqslant 12$ the right-hand side of (4.6) is clearly >0, proving the proposition in those cases. For $r=11$ the right hand side of (4.5) is $\approx 0.4836\ldots$, and so again $t>\frac{1}{4}$. □

4.3. *Proof of Theorem 4.1*

Proof. When $r=9$ we have shown in Corollary 4.4 that v_{α_9} and v_{β_9} are nef; thus we may assume that $r\geqslant 11$.

Suppose that there is a line bundle $L=\mathcal{O}_Y(e_1,e_2)$ with $\alpha_r\leqslant\frac{e_2}{e_1}$ such that $\epsilon_r(L)$ is not equal to the value given in (4.2). Let C be an equal multiplicity curve, either irreducible, or the symmetrization of an irreducible curve as in §1.3, which computes $\epsilon_r(L)$ (or even one which just imposes a stronger condition than imposed by C_3 or C_4).

We have $C^2<0$ and, as explained in §4.2, since C is not equal to any of the C_i, $i=1,\ldots,4$, we also have $C\cdot K_X>0$.

Let L' be the real nef class $\left(\frac{1}{\alpha_r+1},\frac{\alpha_r}{\alpha_r+1}\right)$ (i.e., the first two coordinates of v_{α_r}) on $\mathbb{P}^1\times\mathbb{P}^1$. Since $\alpha_r\leqslant\frac{e_2}{e_1}$, it follows that C must also impose a stronger condition on the Seshadri constant of L' than that imposed by C_3, that is, that

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$$\frac{C_3 \cdot \pi^* L'}{C_3 \cdot E} \geqslant \frac{C \cdot \pi^* L'}{C \cdot E}. \tag{4.7}$$

The reason why is shown in Figure 9 below. Since C must be below $C_3^\perp$ and $C_4^\perp$ it follows that $C^\perp$ must exit the square-zero cone farther to the left than the point where $C_3^\perp$ does, as shown in the picture. Thus $C^\perp$ and $C_3^\perp$ must intersect along a ray, of some slope s. Since C imposes a stronger condition on the Seshadri constant of L than C_3, this means that $\frac{e_2}{e_1} \leqslant s$. But then $\alpha_r \leqslant s$ too, and so similarly C imposes a stronger condition on the Seshadri constant of L' than that imposed by C_3, giving (4.7).

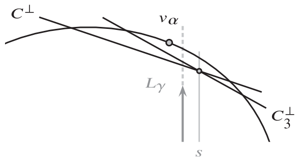

Figure 9. *Graphical argument for (4.7).*

By Proposition 4.2, C has multiplicities $m \geqslant 2$, and is not the symmetrization of a curve C' with all multiplicities in $\{0, 1\}$. Thus we may apply Theorem 4.5 to L', to get

$$\frac{C \cdot \pi^* L'}{C \cdot E} \geqslant \eta_r(L') \cdot \sqrt{\frac{r-t}{r}},$$

with $t = \frac{m-1}{m^2}$. Since $\eta_r(L') = \frac{2}{r}$, combining the previous inequality with (4.7) gives

$$\frac{\frac{r-1}{2}\left(\frac{1}{\alpha_r+1}\right) + 1\left(\frac{\alpha_r}{\alpha_r+1}\right)}{r} = \frac{C_3 \cdot \pi^* L'}{C_3 \cdot E} \geqslant \eta_r(L') \cdot \sqrt{\frac{r-t}{r}} = \frac{2}{r} \cdot \sqrt{\frac{r-t}{r}},$$

or

$$\frac{2\alpha_r + (r-1)}{4(\alpha_r+1)} \geqslant \sqrt{\frac{r-t}{r}}.$$

Applying Proposition 4.7 we conclude that $t > \frac{1}{4}$. But $t = \frac{m-1}{m^2}$, and for $m \geqslant 1$ the maximum value of $\frac{m-1}{m^2}$ is $\frac{1}{4}$, occurring when $m = 2$. This contradiction shows that there can be no such curve C, concluding the proof of Theorem 4.1. □

Remarks 4.8. (1) Here is an outline of the argument above. The final step is that the inequality

$$\frac{C_3 \cdot \pi^* L'}{C_3 \cdot E} \geqslant \eta_r(L') \cdot \sqrt{\frac{r-t}{r}} \tag{4.8}$$

leads to a contradiction whenever $t = \frac{m-1}{m^2}$ with $m \geqslant 1$.

Note that we cannot get (4.8) by applying Theorem 4.5 to C_3, since C_3 does not satisfy the hypothesis of the theorem – all the multiplicities of C_3 are equal to 1. (And it is good that we cannot – the curve C_3 exists, and the inequality leads to a contradiction!)

But, if we assume the existence of a curve C which imposes a stronger condition than C_3 on a line bundle of slope $\geqslant \alpha_r$, we get the inequality (4.7). Applying Theorem 4.5 to C, and combining the inequality which results with (4.7) we arrive at (4.8), and thus a contradiction. Therefore no such curve C can exist.

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(2) As is clear from (4.6), as r gets large, the lower bound estimate on t goes to $\frac{3}{4}$, larger than the $\frac{1}{4}$ needed to give a contradiction. This suggests that one can improve the region on which the formulas in (4.2) hold.

For $s > 0$, let $L(s)$ denote the real nef class of type $(1, s)$ on $\mathbb{P}^1 \times \mathbb{P}^1$. We note that this is different than the L_γ used throughout the rest of the paper. As the idea for the proof of Theorem 4.1 shows, whenever (for a fixed r, r odd, $r \geqslant 9$) s is such that

$$\frac{C_3 \cdot \pi^* L(s)}{r \cdot \eta_r(L(s))} < \sqrt{\frac{r - \frac{1}{4}}{r}}, \tag{4.9}$$

one can conclude that the formulas in (4.2) hold for all line bundles L with slope outside $(\frac{1}{s}, s)$. Solving the inequality (4.9), one finds that the smallest s which works is $s(r) = \frac{1}{4}(2r - \sqrt{12r+1} + 1)$. One has

$$\frac{(\sqrt{r} - 1)^2}{2} < s(r) < \alpha_r.$$

(The leftmost term is the slope where $C_3^\perp$ exits the square-zero cone, see (4.1).) In particular, $(\frac{1}{s(r)}, s(r)) \subseteq (\beta_r, \alpha_r)$.

Thus the previous argument can be used to prove the following result, which, since $(\frac{1}{s(r)}, s(r)) \subseteq (\beta_r, \alpha_r)$, is stronger than Theorem 4.1, and gives an exact value of the Seshadri constant for a narrow range of inner bundles.

Theorem 4.9 (Extension of Theorem 4.1). *Suppose that r is odd, $r \geqslant 9$, that $L = \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(e_1, e_2)$ with $e_1, e_2 \geqslant 1$, and that $\frac{e_2}{e_1} \notin (\frac{1}{s(r)}, s(r))$, where $s(r) = \frac{1}{4}(2r - \sqrt{12r+1} + 1)$. Then*

$$\epsilon_r(L) = \begin{cases} e_2 & \text{if } \frac{e_2}{e_1} \leqslant \frac{2}{r+1}, \\ \frac{2e_1 + (r-1)e_2}{2r} & \text{if } \frac{e_2}{e_1} \in [\frac{2}{r+1}, \frac{1}{s(r)}], \\ \frac{(r-1)e_1 + 2e_2}{2r} & \text{if } \frac{e_2}{e_1} \in [s(r), \frac{r+1}{2}], \\ e_1 & \text{if } \frac{r+1}{2} \leqslant \frac{e_2}{e_1}. \end{cases} \tag{4.10}$$

If one can get better lower bound on the multiplicity of a putative curve C which is K_X-positive and satisfies $C^2 < 0$ (and thus computes the Seshadri constant for some outer bundle), for example, $m \geqslant 3$, $m \geqslant 4$, etc, then one can replace $r - \frac{1}{4}$ in (4.9) by $r - \frac{2}{9}$, $r - \frac{3}{16}$, etc, and get solutions for $s(r)$ which are closer to $\frac{(\sqrt{r}-1)^2}{2}$, and further improve the result above.

Lack of Automorphisms when r is odd. Let us return to the question raised in (2) of Remark 3.2, namely showing that, other than switching the factors, there are no automorphisms of the problem fixing V_r when r is odd.

As we see in Figure 8(a), when r is odd the only nef, outer, square-zero symmetric classes are the fibre classes F_1 and F_2 (this also holds when $r \leqslant 7$, see §5.3–§5.4 below). Thus, any automorphism of the problem either fixes F_1 and F_2, or swaps them. But, if F_1 and F_2 are fixed, in order to preserve the intersection form on V_r, and preserve the nef cone, the automorphism must also fix E, and thus be the identity on V_r. Thus, when r is odd, swapping the factors is the only nontrivial automorphism of the problem.

5. Small r $(1 \leqslant r \leqslant 8)$

In this section we list the results for r between 1 and 8. For $r \leqslant 7$ the blowup of $\mathbb{P}^1 \times \mathbb{P}^1$ at r general points is Fano, and so by Mori's theorem the nef and effective cones are polyhedral. When $r = 8$ the

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anticanonical bundle is nef and effective, and both the nef and effective cones are polyhedral away from the class of $-K_X$. We also note that for $r \leqslant 7$ the numbers α_r and β_r are complex numbers, and so the vectors v_{α_r} and v_{β_r} are not in the real vector space V_r.

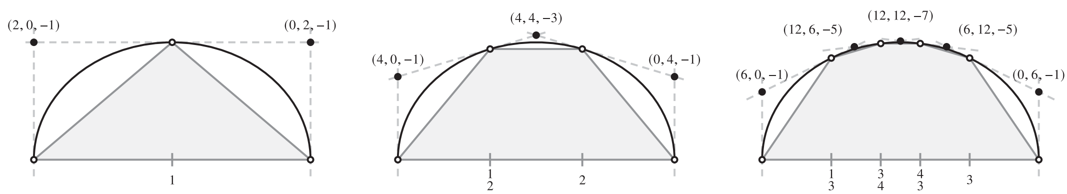
5.1. r = 2, 4, 6

Theorem 3.1 is valid for all even r. In contrast to the cases when $r \geqslant 8$, where T_r has infinite order, T_2, T_4, and T_6 have orders 3, 4, and 6 respectively, and the α_r are the complex roots of unity $\alpha_2 = e^{\frac{2\pi i}{3}}$, $\alpha_4 = i$, and $\alpha_6 = e^{\frac{2\pi i}{6}}$.

Theorem 3.3 showed, when $r \geqslant 10$, r even, that the intersection of the nef cone in V_r and the half plane $K_X^{\leqslant 0}$ is generated by the classes v_{α_r}, v_{β_r} and $T_r^n(F_2)$ with $n \in \mathbb{Z}$, and that the intersection of the effective cone in V_r with the half plane $K_X^{\leqslant 0}$ is generated by v_{α_r}, v_{β_r} and the $T_r^n(E)$, with $n \in \mathbb{Z}$. The arguments in that theorem work here, with the only change being that v_{α_r} and v_{β_r} do not appear at all, and since X is Fano, the intersection with $K_X^{\leqslant 0}$ is all of the nef and effective cones respectively.

Thus, in the cases $r \in \{2, 4, 6\}$, the nef cone is spanned by the classes $T_r^n(F_2)$, for $n = 0, \ldots, \operatorname{ord}(T_r) - 1$ (i.e., to 2, 3, and 5 respectively). Similarly, the effective cones in these cases are spanned by $T_r^n(E)$ for $n = 0, \ldots, \operatorname{ord}(T_r) - 1$.

Figure 10 below shows the nef cones and orbits of these classes. The classes represented by a hollow circle are the orbits of F_2, and the classes represented by a solid circle are the orbits of E (with the exception of the class E itself, which is not shown).



$r = 2$ $r = 4$ $r = 6$

Figure 10. *Cones for* $r = 2, 4,$ *and* 6.

The Seshadri constants for $L = \mathcal{O}_Y(e_1, e_2)$ in these cases are:

$$\epsilon_2(L) = \begin{cases} e_2 & \text{if } \frac{e_2}{e_1} \in (0, 1] \\ e_1 & \text{if } \frac{e_2}{e_1} \in [1, \infty) \end{cases}; \quad \epsilon_4(L) = \begin{cases} e_2 & \text{if } \frac{e_2}{e_1} \in (0, \frac{1}{2}] \\ \frac{e_1+e_2}{3} & \text{if } \frac{e_2}{e_1} \in [\frac{1}{2}, 2] \\ e_1 & \text{if } \frac{e_2}{e_1} \in [2, \infty) \end{cases}; \quad \epsilon_6(L) = \begin{cases} e_2 & \text{if } \frac{e_2}{e_1} \in (0, \frac{1}{3}] \\ \frac{e_1+2e_2}{5} & \text{if } \frac{e_2}{e_1} \in [\frac{1}{3}, \frac{3}{4}] \\ \frac{2e_1+2e_2}{7} & \text{if } \frac{e_2}{e_1} \in [\frac{3}{4}, \frac{4}{3}] \\ \frac{2e_1+e_2}{5} & \text{if } \frac{e_2}{e_1} \in [\frac{4}{3}, 3] \\ e_1 & \text{if } \frac{e_2}{e_1} \in [3, \infty) \end{cases}.$$

5.2. r = 8

When $r = 8$, $\alpha_8 = \beta_8 = 1$, and $v_{\alpha_8} = v_{\beta_8} = -\frac{1}{4}K_X$. The transformation T_8 is unipotent with a single Jordan block.

As in the previous even cases, the argument of Theorem 3.3 shows that the classes $T_8^n(F_2)$, $n \in \mathbb{Z}$, and v_{α_8} $(= v_{\beta_8})$ generate the intersection of the nef cone with $K_X^{\leqslant 0}$. The plane $K_X^{\perp}$ meets the square zero cone (and the nef cone, and the effective cone) only along the ray spanned by v_{α_8}.

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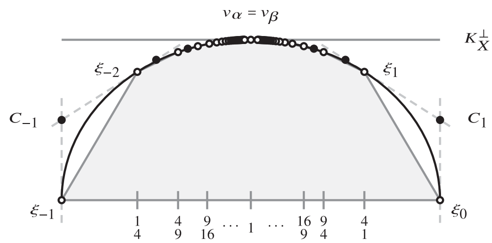

Figure 11. *Cone when $r = 8$.*

As in §3.3, for each $n \in \mathbb{Z}$ we set $\xi_n := T_8^n(F_2)$ and $C_n := T_8^n(E)$. Because T_8 is unipotent, the coordinates of ξ_n and C_n are quadratic functions of n. Specifically,

$$\xi_n = \left(n^2, (n+1)^2, -\binom{n+1}{2}\right) \quad \text{and} \quad C_n = \left(4n(n-1), 4n(n+1), 1-2n^2\right).$$

As in the proof of Theorem 3.3 we have $\xi_{n-1} \cdot C_n = 0 = \xi_n \cdot C_n$ for all $n \in \mathbb{Z}$. Thus the C_n, along with the limiting class v_{α_8}, are dual to the nef cone, and so generate the effective cone of X. The picture in this case is shown in Figure 11 above.

For an ample line bundle $L = \mathcal{O}_Y(e_1, e_2)$ on Y, provided that $\frac{e_2}{e_1} \neq 1$, by the argument in the proof of Corollary 3.4, the Seshadri constant of L is computed by one of the curves C_n above. To find C_n we look for the value of n so that the slope of L is between the slope of ξ_n and the slope of ξ_{n-1}. That is, a value of n so that

$$\frac{(n+1)^2}{n^2} \leqslant \frac{e_2}{e_1} \leqslant \frac{n^2}{(n-1)^2}. \tag{5.1}$$

Here $\frac{1}{0}$ is interpreted as ∞ if necessary, and the conditions imply that $n \geqslant 1$ if $\frac{e_2}{e_1} > 1$ and $n \leqslant -1$ if $\frac{e_2}{e_1} < 1$. For example, C_1 computes the Seshadri constant for those L whose slopes are in $[4, \infty)$, and C_{-1} computes the Seshadri constant for those L whose slopes are in $(0, \frac{1}{4}]$.

Using (5.1), one possible formula for n in these cases is $n = \left\lceil \frac{1}{\sqrt{\frac{e_2}{e_1}}-1} \right\rceil$.

The reason for excluding $\frac{e_2}{e_1} = 1$ is that this is the slope of v_{α_8} (i.e., the limit of the ξ_n, up to scaling, as $n \to +\infty$), and also the slope of v_{β_8} (i.e., the limit of the ξ_n as $n \to -\infty$), and so there are no ξ_n and ξ_{n-1} whose slopes bracket 1. However, v_{α_8} has slope 1, is nef, and lies on the square zero cone. So, for line bundles L of slope 1 the Seshadri constant is the maximum possible value $\epsilon_8(L) = \eta_8(L) = \frac{e_1}{2}$.

In summary,

$$\epsilon_8(L) = \begin{cases} \dfrac{e_1}{2} & \text{if } e_1 = e_2 \\[2ex] \dfrac{n(n+1)(e_1+e_2)}{2(2n^2-1)} & \text{if } e_1 \neq e_2, \text{ with } n = \left\lceil \dfrac{1}{\sqrt{\frac{e_2}{e_1}}-1} \right\rceil \end{cases}.$$

Note that the method of finding n by using φ_r as defined by (1.7) does not work when $r = 8$. Since $\alpha_8 = 1$, the denominator of (1.7) is zero.

5.3. r = 1, 3, 5

Recall that in §4 we defined the curve classes

$$C_1 := (r, 0, -1), \; C_2 := \left(\tfrac{r-1}{2}, 1, -1\right), \; C_3 := \left(1, \tfrac{r-1}{2}, -1\right), \text{ and } C_4 := (0, r, -1), \tag{5.2}$$

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and that for odd $r \geqslant 9$ these classes determine the nef cone for all outer line bundles. When $r < 8$ all ample line bundles on $\mathbb{P}^1 \times \mathbb{P}^1$ are outer (since the plane $K_X^\perp$ does not intersect the square-zero cone). For $r \in \{1, 3, 5\}$ the argument in §4.2 shows that no other symmetric curve class affects the Seshadri constant of an outer bundle (i.e., any ample bundle in this case). Thus, for $r \in \{1, 3, 5\}$ the curve classes above determine the entire nef cone.

One other difference in these cases is that when $r \in \{1, 3\}$ some of these curve classes coincide. Specifically, when $r = 1$ we have $C_3 = C_1$ and $C_2 = C_4$, and when $r = 3$ we have $C_2 = C_3$. The pictures of these curves, and the corresponding nef cones cut out by the half planes $C_i^{\leqslant 0}$, $i = 1, \ldots 4$ is shown below. With the exception of the fibre classes, the small white circles on the square-zero cone are not nef, but are rather classes whose tangent lines contain one of the C_i. As explained in §2.2(b) these classes determine the planes $C_i^\perp$.

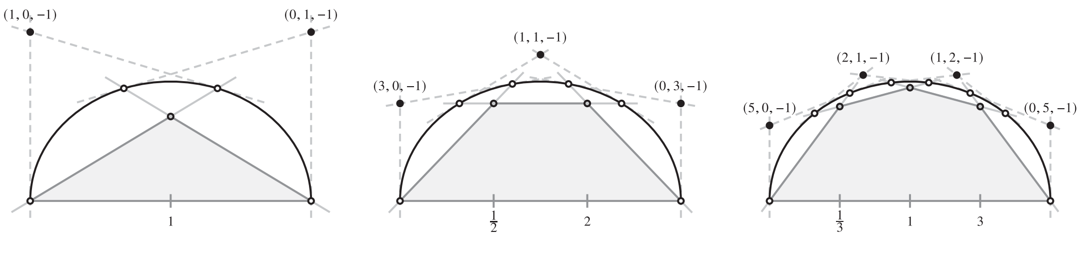


Figure 12. *Cones for* $r = 1, 3, 7$.

For a line bundle $L = \mathcal{O}_Y(e_1, e_2)$ the corresponding Seshadri constants are:

$$\epsilon_1(L) = \begin{cases} e_2 & \text{if } \frac{e_2}{e_1} \in (0, 1] \\ e_1 & \text{if } \frac{e_2}{e_1} \in [1, \infty) \end{cases}; \quad \epsilon_3(L) = \begin{cases} e_2 & \text{if } \frac{e_2}{e_1} \in (0, \frac{1}{2}] \\ \frac{e_1+e_2}{3} & \text{if } \frac{e_2}{e_1} \in [\frac{1}{2}, 2] \\ e_1 & \text{if } \frac{e_2}{e_1} \in [2, \infty) \end{cases}; \quad \epsilon_5(L) = \begin{cases} e_2 & \text{if } \frac{e_2}{e_1} \in (0, \frac{1}{3}] \\ \frac{e_1+2e_2}{5} & \text{if } \frac{e_2}{e_1} \in [\frac{1}{3}, 1] \\ \frac{2e_1+e_2}{5} & \text{if } \frac{e_2}{e_1} \in [1, 3] \\ e_1 & \text{if } \frac{e_2}{e_1} \in [3, \infty) \end{cases}.$$

It is interesting to note the similarities between the formulas above and those in §5.1.

5.4. r = 7

As in §5.3 all ample bundles on $\mathbb{P}^1 \times \mathbb{P}^1$ are outer. The main difference in the case $r = 7$ from the cases of all other odd r is that the Seshadri constants of outer bundles (i.e., all ample bundles in this case) are determined by five curve classes. In addition to the four curve classes in (5.2) there is an additional class, which for reasons of consistency in the diagram we label $C_{2\frac{1}{2}}$:

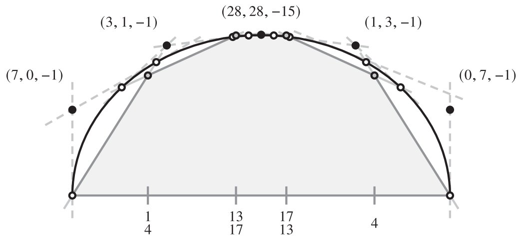
$$C_{2\frac{1}{2}} := (28, 28, -15).$$

This curve class is the union of 7 disjoint (-1)-classes. The curve class $C' = 4F_1 + 4F_2 - E - E_1$ (i.e., bidegree $(4, 4)$, multiplicity 2 at p_1, and multiplicity 1 at $p_2, \ldots, p_7$) satisfies $(C')^2 = -1$ and $C' \cdot K_X = -1$. Starting with the linear series $|\mathcal{O}_Y(2, 2)|$, and imposing multiplicity 2 at a point p_1, the general member of the resulting linear series is irreducible. Imposing the condition that the linear series pass through six further general points, we conclude that the class C' is represented by an irreducible (-1)-curve. The symmetrization of C', as in §1.3, is the class $C_{2\frac{1}{2}}$.

The argument in §4.2, used again in §5.3, which shows that $C_1, \ldots, C_4$ determine the Seshadri constant of all outer bundles is the following. Each time one finds a (-1)-curve class, or a class which is a symmetrization of (-1)-curves, one draws the corresponding half-plane $C_i^{\leqslant 0}$. Any subsequent such class has to lie in that half plane. When $r \in \{1, 3, 5\}$ the half planes corresponding to $C_1, \ldots, C_4$

eliminate the possibility of any other curve class with negative self-intersection. When $r \geqslant 9$ these half planes do not eliminate the possibility of any other curve class with negative self-intersection, but do eliminate the possibility of any curve class with negative intersection which is K_X-negative. The case $r = 7$ is intermediate between these behaviours. Here the classes $C_1, \ldots, C_4$ do not eliminate all classes with negative self-intersection, not even in the half plane $K_X^{\leqslant 0}$. However, the addition of the new curve class $C_{2\frac{1}{2}}$ to the list is sufficient to rule out all further possibilities.

The picture in the case $r = 7$, along with the corresponding Seshadri constants for a line bundle $L = \mathcal{O}_Y(e_1, e_2)$, appear below.



$$\epsilon_7(L) = \begin{cases} e_2 & \text{if } \frac{e_2}{e_1} \in (0, \frac{1}{4}] \\ \frac{e_1+3e_2}{7} & \text{if } \frac{e_2}{e_1} \in [\frac{1}{4}, \frac{13}{17}] \\ \frac{4e_1+4e_2}{15} & \text{if } \frac{e_2}{e_1} \in [\frac{13}{17}, \frac{17}{13}] \\ \frac{3e_1+e_2}{7} & \text{if } \frac{e_2}{e_1} \in [\frac{17}{13}, 4] \\ e_1 & \text{if } \frac{e_2}{e_1} \in [4, \infty) \end{cases}.$$

Figure 13. *Cone for $r = 7$.*

Remark 5.1. The blowup of $\mathbb{P}^1 \times \mathbb{P}^1$ at r general points is the same as the blowup of $\mathbb{P}^2$ at $(r+1)$ general points. The effective cone, and the generating (-1)-curves, for $\mathbb{P}^2$ blown up at $\leqslant (7+1)$ points are well known. For example, the generators are listed in [9, p. 135, Proposition 26.1].

Applying the change of basis formula between the Picard group of $\mathbb{P}^2$ blown up at $(r+1)$ points, and the blowup of $\mathbb{P}^1 \times \mathbb{P}^1$ at r points, one obtains generators for the effective cones of $\mathbb{P}^1 \times \mathbb{P}^1$ blown up at $r \leqslant 7$ points. These (-1)-curves, or their symmetrizations, give generators of the symmetric effective cones in §5.1, 5.3, and 5.4. This is another way to arrive at the description of the cones given above. Note that not all (-1)-curves, or their symmetrizations, appear as boundary generators of the symmetrized effective cone. Any class which does appear has to satisfy some restrictive conditions on its multiplicities, see [4, Theorem 2.6.2].

6. A brief study of the slopes of the ξ_n

For fixed r, r even, recall that in §1.7 we have defined sequences $\{q_{n,r}\}_{n\in\mathbb{Z}}$ by setting $q_{-1,r} = 1$, $q_{0,r} = 0$, $q_{1,r} = 1$, and then using the recursion (1.9). For instance, from the recursion, $q_{2,r} = \frac{r-2}{2}(q_{1,r} - q_{0,r}) + q_{-1,r} = \frac{r}{2}$.

In the proof of Theorem 3.3 we have defined classes $\xi_n = \xi_{n,r}$ for all $n \in \mathbb{Z}$ by $\xi_{n,r} = T_r^n(F_2)$ (in the proof the dependence on r was omitted from the notation). Thus in the usual coordinates on V_r,

$$\begin{aligned} \xi_{-1,r} &= (1,0,0) = \left(q_{-1,r}, q_{0,r}, -\sqrt{\frac{2q_{-1,r}q_{1,r}}{r}}\right), \\ \xi_{0,r} &= (0,1,0) = \left(q_{0,r}, q_{1,r}, -\sqrt{\frac{2q_{0,r}q_{1,r}}{r}}\right), \text{ and} \\ \xi_{1,r} &= (1, \tfrac{r}{2}, -1) = \left(q_{1,r}, q_{2,r}, -\sqrt{\frac{2q_{1,r}q_{2,r}}{r}}\right). \end{aligned}$$

We note that the third coordinate is determined by the first two, since by Theorem 3.1 $\xi_{n,r}^2 \overset{3.1}{=} F_2^2 = 0$ for each n. Since the recursion (1.9) is the one given by the characteristic polynomial of T_r, we conclude that

$$\xi_{n,r} = \left(q_{n,r}, q_{n+1,r}, -\sqrt{\frac{2q_{n,r}q_{n+1,r}}{r}}\right) \text{ for all } n \in \mathbb{Z}. \tag{6.1}$$

In particular, the slope of $\xi_{r,n}$ is $\frac{q_{n+1,r}}{q_{n,r}}$.

In this section we prove a few results about these slopes for use in §7 below, as well as demonstrate the properties of the map φ_r defined in (1.8).

Lemma 6.1. *Let $r \geqslant 8$ be even. Then the sequence of slopes $\{\frac{q_{n+1,r}}{q_{n,r}}\}_{n\geqslant 1}$ is strictly decreasing.*

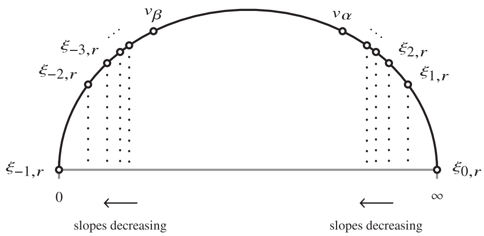


Figure 14. *Slopes of the ξ_n.*

Proof. When $r = 8$, from §5.2 we have the explicit formula $q_n = n^2$, which immediately shows that the sequence is decreasing. When $r \geqslant 10$, the argument is that it "follows from the picture". As shown in Figure 14, the slopes of $\xi_{1,r}, \xi_{2,r}, \xi_{3,r}, \ldots$ are decreasing. The figure also shows that the slopes on the other side are decreasing. That is, restricted to the set $\{n \in \mathbb{Z} \mid n \leqslant -1\}$, the function $n \mapsto \frac{q_{n+1,r}}{q_{n,r}}$ is decreasing. This second statement also holds when $r = 8$, by the explicit formula above. □

The only point where the sequence of slopes $\{\frac{q_{n+1,r}}{q_{n,r}}\}_{n\in\mathbb{Z}}$ fails to be decreasing is the transition from $n = -1$ (where the slope is 0) to $n = 0$ (where the slope is ∞). To justify that this picture is always correct, and thus the argument of the proof is correct, we use the properties of φ_r, exposed below. The justification of the picture appears in §6.3.

6.1. Properties of T_r^s

It is convenient to define T_r^s for all $s \in \mathbb{R}$, and not only $s \in \mathbb{Z}$. This is possible since T_r is diagonalizable, and all eigenvalues are positive real numbers. Specifically, when $r \geqslant 10$ the vectors v_{α_r}, v_{β_r}, and v_1 of (1.6) are a basis of eigenvectors of V_r, and we define T_r^s on V_r by setting

$$T_r^s(v_{\alpha_r}) = \alpha_r^s\, v_{\alpha_r},\ T_r^s(v_\beta) = \beta_r^s\, v_\beta, \quad \text{and} \quad T_r^s(v_1) = 1^s\, v_1 = v_1. \tag{6.2}$$

The intersections in Table 1 and the fact that $\alpha_r \cdot \beta_r = 1$ show that T_r^s preserves the intersection form on V_r. By construction T_r^s also fixes v_1, the class of K_X.

6.2. Properties of φ_r

Suppose that $v \in V_r$, and write v as a linear combination of the basis vectors above:

$$v = a\, v_{\alpha_r} + b\, v_{\beta_r} + c\, v_1. \tag{6.3}$$

Assuming that $b \neq 0$, and using the intersections in Table 1 we compute that

$$\frac{v \cdot v_{\beta_r}}{v \cdot v_{\alpha_r}} = \frac{a}{b}.$$

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For each $s \in \mathbb{R}$ let $v_s = T_r^s(v)$ (so $v_0 = v$). Then by (6.2) $v_s = a\alpha_r^s\, v_{\alpha_r} + b\beta_r^s\, v_{\beta_r} + c\, v_1$, and so

$$\frac{v_s \cdot v_{\beta_r}}{v_s \cdot v_{\alpha_r}} = \frac{a\alpha_r^s}{b\beta_r^s} = \frac{a}{b} \cdot \alpha_r^{2s}.$$

If, in addition, $a \neq 0$ then

$$\log\Big(\tfrac{v_s \cdot v_{\beta_r}}{v_s \cdot v_{\alpha_r}}\Big) = \log\big(\tfrac{a}{b}\big) + 2s\log(\alpha_r) = \log\Big(\tfrac{v \cdot v_{\beta_r}}{v \cdot v_{\alpha_r}}\Big) + 2s\log(\alpha_r),$$

from which we conclude that

$$\varphi_r(v_s) = \varphi_r(v) + s. \tag{6.4}$$

Now suppose that v is on the square-zero cone. Then, in the coordinates from (6.3), $v^2 = \frac{r-8}{r}(2ab - rc^2) = 0$. If in addition $ab = 0$ we conclude that $c = 0$, and that v is a multiple of either v_{α_r} or v_{β_r}. Conversely, if v is such that $v^2 = 0$ and is not a multiple of v_{α_r} or v_{β_r}, then $ab \neq 0$, which shows that the formula in (1.8) is well defined.

As already noted in §1.4, for all $\lambda \in \mathbb{R}^*$, $\varphi_r(\lambda v) = \varphi_r(v)$. Thus φ_r is a well-defined function on the square-zero cone minus the lines spanned by v_{α_r} and v_{β_r}, modulo scaling by $\mathbb{R}^*$.

6.3. Applications to the square-zero cone

When $r \geqslant 10$ the plane $K_X^{\perp}$ intersects the square-zero cone transversely (away from zero). Thus, the quotient above consists of two open arcs, each homeomorphic to $\mathbb{R}$.

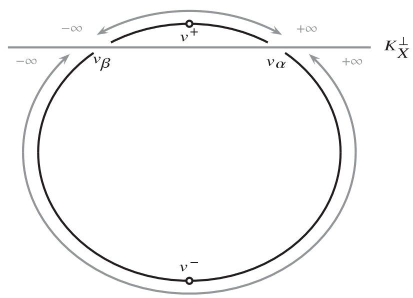

Figure 15. *Action of T_r^s, and coordinates given by φ_r.*

If $v^2 = 0$ then $T_r^s(v)^2 = 0$, since T_r^s preserves the intersection form. Similarly, since T_r^s preserves K_X, if v is on the arc in $K_X^{>0}$ (respectively in $K_X^{<0}$) then so is $T_r^s(v)$.

If v is on one of the arcs then as observed above, in the coordinates from (6.3), $ab \neq 0$, and thus (modulo scaling) as $s \to \infty$, $T_r^s(v) \to v_{\alpha_r}$ and as $s \to -\infty$, $T_r^s(v) \to v_{\beta_r}$.

If v is as above, and so $ab \neq 0$, the formula for the action of T_r^s shows that $T_r^s(v)$ is a scalar multiple of v if and only if $s = 0$. Thus, the action of the group $(\mathbb{R}, +)$ on each of the upper and lower arcs, where $s \in \mathbb{R}$ acts on v via $s \cdot v = T_r^s(v)$, is simply transitive.

Setting $v^{\pm} = (1, 1, \pm\sqrt{\tfrac{2}{r}})$, v^+ is on the upper arc, and v^- is on the lower arc. The maps $s \to T_r^s(v^+)$ and $s \to T_r^s(v^-)$ are therefore continuous bijections of $\mathbb{R}$ with the upper and lower arcs respectively. Since $\varphi_r(v^{\pm}) = 0$, (6.4) shows that φ_r provides a continuous inverse to each of the previous bijections. This allows us to put coordinates on the upper and lower arcs. The situation is illustrated in Figure 15 above.

Justification of the picture used in the proof of Lemma 6.1. Since $\xi_{n,r} = T_r^n(\xi_{0,r}) = T_r^n(F_2)$ for all $n \in \mathbb{Z}$, each $\xi_{n+1,r}$ is farther along the lower arc in the positive direction than $\xi_{n,r}$, where the notion of

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positive is provided by the action of $(\mathbb{R}, +)$ above. Given the locations of ξ_{-1} and ξ_0 on the square zero cone, this shows that the picture in Figure 14 is correct, and hence the deduction from it used in the proof of Lemma 6.1 is also correct.

We now return to studying the slopes of the ξ_n.

Definition 6.2. For each positive integer m, define $J_m \subseteq \mathbb{R}$ by

$$J_m := \left(m - \tfrac{1}{2}, m\right) \cup \left(m, m + \tfrac{1}{2}\right) \cup \{m+2\} = \left(\left(m - \tfrac{1}{2}, m + \tfrac{1}{2}\right) \setminus \{m\}\right) \cup \{m+2\}.$$

We note that if $m \neq m'$, then $J_m \cap J_{m'} = \varnothing$.

Proposition 6.3. *Let r be even, $r \geqslant 10$. Then the sequence $\{\frac{q_{n+1,r}}{q_{n,r}}\}_{n\geqslant 1}$ is contained in $J_{\frac{r-4}{2}}$.*

Proof. Starting with $q_{-1,r} = 1$, $q_{0,r} = 0$, $q_{1,r} = 1$, and using (1.9) we compute that

$$q_{2,r} = \tfrac{r}{2}, \ q_{3,r} = \tfrac{(r-2)^2}{4}, \ \text{and} \ q_{4,r} = \tfrac{r(r-4)^2}{8},$$

which then gives

$$\tfrac{q_{2,r}}{q_{1,r}} = \tfrac{r}{2}, \ \tfrac{q_{3,r}}{q_{2,r}} = \tfrac{r-4}{2} + \tfrac{2}{r}, \ \text{and} \ \tfrac{r-4}{2} - \tfrac{q_{4,r}}{q_{3,r}} = \tfrac{2r-8}{(r-2)^2}.$$

Since $\frac{2r-8}{(r-2)^2} > 0$, we conclude that $\frac{q_{4,r}}{q_{3,r}} < \frac{r-4}{2}$. As $n \to \infty$, $\frac{q_{n+1,r}}{q_{n,r}} \to \alpha_r$. By Lemma 6.1 the sequence $\{\frac{q_{n+1,r}}{q_{n,r}}\}_{n\geqslant 1}$ is decreasing. Thus the previous calculations show that $\{\frac{q_{n+1,r}}{q_{n,r}}\}_{n\geqslant 1}$ is contained in the set

$$\left(\alpha_r, \tfrac{r-4}{2}\right) \cup \left\{\tfrac{r-4}{2} + \tfrac{2}{r}\right\} \cup \left\{\tfrac{r}{2}\right\}. \tag{6.5}$$

Lemma 4.6 gives the estimate $\frac{r-5}{2} \leqslant \alpha_r$, and so the set in (6.5) is contained in the set

$$\left(\tfrac{r-5}{2}, \tfrac{r-4}{2}\right) \cup \left\{\tfrac{r-4}{2} + \tfrac{2}{r}\right\} \cup \left\{\tfrac{r}{2}\right\}. \tag{6.6}$$

In turn, for $m = \frac{r-4}{2}$, the set in (6.6) is contained in J_m, proving the proposition. □

Theorem 6.4 (Uniqueness of r). *Let e_1 and e_2 be positive integers. Then there is at most one even r, $r \geqslant 2$, such that $\frac{e_2}{e_1} = \frac{q_{n+1,r}}{q_{n,r}}$ for some $n \in \mathbb{Z}$. If $r \geqslant 8$ then the value of n is also unique.*

Proof. By symmetry we may restrict to the case $\frac{e_2}{e_1} \geqslant 1$. If $\frac{e_2}{e_1} = \frac{q_{n+1,r}}{q_{n,r}}$ and $r \geqslant 8$, this implies that $n \geqslant 1$. For $r = 2$, 4, and 6 the sequence is periodic; see §5.1.

By Proposition 6.3 for each even $r \geqslant 10$ the slopes $\{\frac{q_{n+1,r}}{q_{n,r}}\}_{n\geqslant 1}$ are contained in $J_{\frac{r-4}{2}}$. Since $J_m \cap J_{m'} = \varnothing$ if $m \neq m'$, the slopes for different r, $r \geqslant 10$ do not coincide.

The possible slopes ($\geqslant 1$) when $r = 2$, 4, 6, and 8 are: 1 ($r = 2$); 2 ($r = 4$); $\frac{4}{3}$ and 6 ($r = 6$); and $\left\{\frac{(n+1)^2}{n^2}\right\}_{n\geqslant 1}$ ($r = 8$). See §5.1–§5.2.

These slopes are distinct, and none are contained in the sets J_m for $m \geqslant 3$. This proves uniqueness of r.

If $r \geqslant 8$ the uniqueness of n follows from the fact that, by Lemma 6.1, the sequence of slopes $\{\frac{q_{n+1,r}}{q_{n,r}}\}_{n\geqslant 1}$ is strictly decreasing. □

Remarks 6.5. (1) Not all positive rational numbers are the slopes of some $\xi_{n,r}$, i.e., there are many slopes $\frac{e_2}{e_1}$ which are not of the form $\frac{q_{n+1,r}}{q_{n,r}}$ for some n and r. For instance, when $m \geqslant 3$ the only rational number in $(m, m + \frac{1}{2})$ which is a slope of some $\xi_{n,r}$ is $\frac{(m+1)^2}{(m+2)}$ (i.e., $\frac{r-4}{2} + \frac{2}{r}$, where $r = 2m+4$), the slope of $\xi_{2,2m+4}$.

(2) The description above of the slopes, particularly the partition of the slopes into the sets J_m, can be used to give an algorithm to decide whether a particular slope $\frac{e_2}{e_1}$ is the slope of some $\xi_{n,r}$.

7. Other arguments related to the symplectic packing problem

In this section we use the results from §3–§6 to prove Theorems 1.2 and 1.3. We also give the formulae for the packing constants when $r \leqslant 8$. We start by recalling the dictionary between the differential-geometric language and the algebro-geometric one.

If M is the real manifold underlying $\mathbb{P}^1 \times \mathbb{P}^1$, and ω_M a symplectic form on M, then the cohomology class of ω_M is an element of $H^2(M, \mathbb{R}) \cong \mathbb{R}^2$, with a natural basis coming from the Künneth theorem. In this basis the cohomology class of ω_M corresponds to a pair (e_1, e_2). By [8, Theorem 1.1], up to diffeomorphism of M, every such form ω_M comes from a Kähler form on $\mathbb{P}^1 \times \mathbb{P}^1$, and for such a form $e_1, e_2 > 0$. It is these numbers in the formulas below which give the packing constant associated to ω_M.

We also note, although we will not need to use this, that the packing constant is homogeneous of degree 0 in e_1 and e_2, and so unchanged by simultaneous scaling. This can be seen directly from the definition of the packing constant, or, in our case, from the formulae in Theorem 1.2 and in §7.2 below.

Recall that by the theorem of Biran [1, Theorem 6.A] for $Y = \mathbb{P}^1 \times \mathbb{P}^1$ and L a real ample class of type (e_1, e_2) (i.e., $e_1, e_2 \in \mathbb{R}$, $e_1, e_2 > 0$) one has

$$\nu_r(L) = \left(\frac{\tilde{\epsilon}_r(L)}{\eta_r(L)}\right)^2 = \frac{r(\tilde{\epsilon}_r(L))^2}{2e_1e_2}. \tag{7.1}$$

Here $\tilde{\epsilon}_r(L)$ is like the Seshadri constant but restricting the test curves to be (-1)-curves or, equivalently, their symmetrizations (see (1.12) for the definition of $\tilde{\epsilon}_r(L)$).

In the calculations in §3–§5 we have found all the (-1)-curves, or symmetrizations of (-1)-curves, which can affect a Seshadri constant, and thus can compute $\tilde{\epsilon}_r(L)$ for all r. Reversing these formulae we find the conditions on r for when L of type (e_1, e_2) admits a full-packing.

7.1. *Proof of Theorem 1.2*

The case of odd r. In §4 we have shown that the curve classes $C_1, \ldots, C_4$ are the only (-1)-curves, or symmetrizations of such, which affect Seshadri constants when $r \geqslant 9$ is odd.

The curves $C_1, \ldots, C_4$ affect the Seshadri constants of bundles of slopes $(0, \frac{2}{r+1}]$, $[\frac{2}{r+1}, \frac{2}{(\sqrt{r}-1)^2}]$, $[\frac{(\sqrt{r}-1)^2}{2}, \frac{r+1}{2}]$, and $[\frac{r+1}{2}, \infty)$ respectively.

In §4 we were unable to show that C_2 and C_3 computed the Seshadri constant over their entire respective intervals, and instead restricted ourselves to smaller intervals where we could justify this. However, for $\tilde{\epsilon}_r(L)$ there are no other competing curve classes to consider. Thus, $\tilde{\epsilon}_r(L)$ is computed by the curves $C_1, \ldots, C_4$ on the respective intervals listed above, and is equal to $\nu_r(L)$ on the intermediate interval $[\frac{2}{(\sqrt{r}-1)^2}, \frac{(\sqrt{r}-1)^2}{2}]$. This gives (1.13).

The case of even r. In the proof of Theorem 3.3 we showed that the only (-1)-curves (or symmetrizations) which affected the Seshadri constants of line bundles were the curve classes $C_n = C_{n,r} = T_r^n(E)$, and that these curve classes computed the Seshadri constants for outer bundles (i.e., bundles whose slope is outside of $[\beta_r, \alpha_r]$) and did not affect any inner bundles (bundles whose slope is in $[\beta_r, \alpha_r]$). Thus $\tilde{\epsilon}_r(L) = \epsilon_r(L)$ for outer bundles, and for inner bundles $\tilde{\epsilon}_r(L) = \eta_r(L)$. This gives (1.14).

7.2. *Packing constants for* $r \leqslant 8$

In §5 we computed the Seshadri constants when $r \leqslant 8$ for all ample L, and all Seshadri constants were computed by (-1)-curves or their symmetrizations. Thus, for $r \leqslant 8$ we have $\tilde{\epsilon}_r(L) = \epsilon_r(L)$ for all ample L. By (7.1), for L a real ample bundle of type (e_1, e_2), we therefore have $\nu_r(L) = \frac{r(\epsilon_r(L))^2}{2e_1e_2}$ for all $r \leqslant 8$. For convenience we list the formulae for those packing constants here.

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$$\nu_1(L) = \begin{cases} \frac{e_2}{2e_1} & \textit{if } \frac{e_2}{e_1} \in (0,1] \\ \frac{e_1}{2e_2} & \textit{if } \frac{e_2}{e_1} \in [1,\infty) \end{cases}; \quad \nu_3(L) = \begin{cases} \frac{3e_2}{2e_1} & \textit{if } \frac{e_2}{e_1} \in (0,\frac{1}{2}] \\ \frac{(e_1+e_2)^2}{6e_1e_2} & \textit{if } \frac{e_2}{e_1} \in [\frac{1}{2},2] \\ \frac{3e_1}{2e_2} & \textit{if } \frac{e_2}{e_1} \in [2,\infty) \end{cases}; \quad \nu_5(L) = \begin{cases} \frac{5e_2}{2e_1} & \textit{if } \frac{e_2}{e_1} \in (0,\frac{1}{3}] \\ \frac{(e_1+2e_2)^2}{10e_1e_2} & \textit{if } \frac{e_2}{e_1} \in [\frac{1}{3},1] \\ \frac{(2e_1+e_2)^2}{10e_1e_2} & \textit{if } \frac{e_2}{e_1} \in [1,3] \\ \frac{5e_1}{2e_2} & \textit{if } \frac{e_2}{e_1} \in [3,\infty) \end{cases};$$

$$\nu_2(L) = \begin{cases} \frac{e_2}{e_1} & \textit{if } \frac{e_2}{e_1} \in (0,1] \\ \frac{e_1}{e_2} & \textit{if } \frac{e_2}{e_1} \in [1,\infty) \end{cases}; \quad \nu_4(L) = \begin{cases} \frac{2e_2}{e_1} & \textit{if } \frac{e_2}{e_1} \in (0,\frac{1}{2}] \\ \frac{2(e_1+e_2)^2}{9e_1e_2} & \textit{if } \frac{e_2}{e_1} \in [\frac{1}{2},2] \\ \frac{2e_1}{e_2} & \textit{if } \frac{e_2}{e_1} \in [2,\infty) \end{cases}; \quad \nu_6(L) = \begin{cases} \frac{3e_2}{e_1} & \textit{if } \frac{e_2}{e_1} \in (0,\frac{1}{3}] \\ \frac{3(e_1+2e_2)^2}{25e_1e_2} & \textit{if } \frac{e_2}{e_1} \in [\frac{1}{3},\frac{3}{4}] \\ \frac{12(e_1+e_2)^2}{49e_1e_2} & \textit{if } \frac{e_2}{e_1} \in [\frac{3}{4},\frac{4}{3}] \\ \frac{3(2e_1+e_2)^2}{25e_1e_2} & \textit{if } \frac{e_2}{e_1} \in [\frac{4}{3},3] \\ \frac{3e_1}{e_2} & \textit{if } \frac{e_2}{e_1} \in [3,\infty) \end{cases};$$

$$\nu_7(L) = \begin{cases} \frac{7e_2}{2e_1} & \textit{if } \frac{e_2}{e_1} \in (0,\frac{1}{4}] \\ \frac{(e_1+3e_2)^2}{14e_1e_2} & \textit{if } \frac{e_2}{e_1} \in [\frac{1}{4},\frac{13}{17}] \\ \frac{56(e_1+e_2)^2}{225e_1e_2} & \textit{if } \frac{e_2}{e_1} \in [\frac{13}{17},\frac{17}{13}] \\ \frac{(3e_1+e_2)^2}{14e_1e_2} & \textit{if } \frac{e_2}{e_1} \in [\frac{17}{13},4] \\ \frac{7e_1}{2e_2} & \textit{if } \frac{e_2}{e_1} \in [4,\infty) \end{cases}; \quad \text{and} \quad \nu_8(L) = \begin{cases} 1 & \textit{if } e_1 = e_2 \\ \frac{n^2(n+1)^2(e_1+e_2)^2}{(2n^2-1)^2e_1e_2} & \textit{if } e_1 \neq e_2, \textit{ with } n = \left\lceil \frac{1}{\sqrt{\frac{e_2}{e_1}}-1} \right\rceil \end{cases}.$$

7.3. Proof of Theorem 1.3

By (7.1) one has a full packing if and only if $\tilde{\epsilon}_r(L) = \eta_r(L)$. In terms of our graphical description of the Seshadri constants (c.f. §2.2(c) or Figure 7) this means that the bundle L_γ, with $\gamma = \tilde{\epsilon}_r(L)$ has reached the square-zero cone without crossing any plane of the form $C^\perp$, where C is a (-1)-curve or symmetrization of a (-1)-curve. If we are in a region where the Seshadri constant is computed by such curves, this means that L_γ is a nef class on the square-zero cone.

The case of odd r. In §5.3–§5.4 we have seen that for r odd, $r \leqslant 7$ (where all Seshadri constants are determined by (-1)-curves or their symmetrizations) there are no nef square-zero classes. In other words, in the pictures in §5.3–§5.4, the nef cone never reaches the square-zero cone, although this is a bit difficult to see in the picture for $r = 7$. Thus, to have a full packing when r is odd, one needs at least $r \geqslant 9$.

When $r \geqslant 9$ and is odd we have seen in §4, that the curve classes $C_1, \ldots, C_4$ only affect Seshadri constants for bundles with slopes outside $[\frac{2}{(\sqrt{r}-1)^2}, \frac{(\sqrt{r}-1)^2}{2}]$, and that outside that interval the nef cone never reaches the square-zero cone.

Thus, when r is odd a full packing occurs for a real line bundle of type (e_1, e_2) if and only if $r \geqslant 9$ and $\frac{2}{(\sqrt{r}-1)^2} \leqslant \frac{e_2}{e_1} \leqslant \frac{(\sqrt{r}-1)^2}{2}$. This is equivalent to the condition

$$r \geqslant \max\left(\left(\sqrt{\frac{2e_2}{e_1}}+1\right)^2, \left(\sqrt{\frac{2e_1}{e_2}}+1\right)^2, 9\right)$$

of Theorem 1.3.

The case of even r. In the proof of Theorem 3.3, when r is even, $r \geqslant 10$, we have seen that the curve classes $C_{n,r} := T_r^n(E)$, each the union of r disjoint (-1)-curves, determine the Seshadri constants for all

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outer bundles, i.e., bundles whose slope is outside $[\beta_r, \alpha_r]$, and that no (-1)-curve or symmetrization affects the Seshadri constant of inner bundles.

We have also seen that when $r \geqslant 10$ the classes $\xi_{n,r}$ are the only nef classes on the square-zero cone with slope outside of $[\beta_r, \alpha_r]$. In §5, when r is even, $r \leqslant 8$, we have similarly seen that the classes $\xi_{n,r}$ (and v_{α_8}) are the only nef classes on the square-zero cone.

Thus, when r is even a full packing occurs for a real line bundle of type (e_1, e_2) if and only if

(i) $\beta_r \leqslant \frac{e_2}{e_1} \leqslant \alpha_r$, or
(ii) r is a value for which $\frac{e_2}{e_1}$ is equal to the slope of a $\xi_{n,r}$ for some n.

These are equivalent to the conditions

(i) $r \geqslant \frac{2(e_1+e_2)^2}{e_1e_2}$, or
(ii) r is a value for which $\frac{e_2}{e_1}$ is equal to $\frac{q_{n+1,r}}{q_{n,r}}$ for some n,

appearing in Theorem 1.3.

In addition, we recall that by Theorem 6.4, for a fixed (e_1, e_2) there is at most one value of r where condition (*ii*) occurs. This completes the arguments related to the symplectic packing problem.

8. Petrakiev reflections

8.1. The reflection theorem

A common technique for investigating the cone of effective classes on a surface blown up at r general points is to specialize the points to lie on some fixed curve G in such a way that the proper transform G_0 of G has negative self-intersection. If C is a class which is effective when the points are in general position, then under the condition that $G_0 \cdot G_0 < 0$ the class $C - (C \cdot G_0)/(G_0 \cdot G_0)G_0$ is effective when the points are in special position, and this can be used to deduce restrictions on C.

A beautiful argument of I. Petrakiev allows one, under mild conditions on G and r, to double the coefficient of G_0 subtracted in the formula above. The formula then becomes that for the reflection of C in G_0. Dually, one may reflect classes which are nef on the specialization to get classes which are nef when the points are in general position. In this section we record this result, and in the next use it to produce inner nef classes on the square-zero cone for even and odd r.

Theorem 8.1 (Reflection Theorem). *Let Y be a smooth surface, and $G \subset Y$ a smooth irreducible curve. We use X to denote the blowup of Y at r general points, and X_0 to denote the blowup of Y at r general points of G. We identify the Néron-Severi groups of X and X_0, along with their intersection forms, via the isomorphisms*

$$\mathrm{NS}(X)_{\mathbb{Q}} \cong \mathrm{NS}(Y) \bigoplus \oplus_{i=1}^{r} \mathbb{Q}E_i \cong \mathrm{NS}(X_0)_{\mathbb{Q}},$$

and use V to denote the common inner product space. We additionally assume the following numerical conditions: that $r \geqslant |G \cdot G|$; that $(G \cdot G) - r < 0$ (thus $r = |G \cdot G|$ is only allowed when $G \cdot G < 0$); and, if G has genus 0, that $(G \cdot G) - r$ is even. We denote by G_0 the proper transform of G in X_0 (therefore $G_0 \cdot G_0 = (G \cdot G) - r < 0$), and by $\varphi_{G_0} : V \longrightarrow V$ the isometry "reflection in G_0" given by the formula

$$\varphi_{G_0}(\xi) := \xi - 2\left(\frac{\xi \cdot G_0}{G_0 \cdot G_0}\right)G_0. \tag{8.1}$$

Under the numerical conditions above,

(a) *If C is an effective class on X, then $\varphi_{G_0}(C)$ is an effective class on X_0.*
(b) *If ξ_0 is a nef class on X_0, then $\xi := \varphi_{G_0}(\xi_0)$ is a nef class on X, and $\xi \cdot \xi = \xi_0 \cdot \xi_0$.*

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Proof. We first show that (a) implies (b). We recall that

(i) φ_{G_0} is an isometry, that is, $\varphi_{G_0}(\xi_1)\cdot\varphi_{G_0}(\xi_2)=\xi_1\cdot\xi_2$ for all $\xi_1,\xi_2\in V$, and
(ii) φ_{G_0} is self-adjoint, that is, $\xi_1\cdot\varphi_{G_0}(\xi_2)=\varphi_{G_0}(\xi_1)\cdot\xi_2$ for all $\xi_1,\xi_2\in V$.

(The two statements are equivalent, since $\varphi_{G_0}^2=\mathrm{Id}_V$.)

If C is an effective class on X, then by (a), $\varphi_{G_0}(C)$ is an effective class on X_0, and thus, since ξ_0 is a nef class on X_0, $\varphi_{G_0}(C)\cdot\xi_0\geqslant 0$. Therefore

$$C\cdot\xi=C\cdot\varphi_{G_0}(\xi_0)\overset{(ii)}{=\!=}\varphi_{G_0}(C)\cdot\xi_0\geqslant 0.$$

Since $C\cdot\xi\geqslant 0$ for all effective classes C on X we conclude that ξ is a nef class on X. Furthermore, $\xi\cdot\xi=\xi_0\cdot\xi_0$ since φ_{G_0} is an isometry.

Part (a) is proved in [11], although the result is not explicitly stated in that form. We will outline the argument, indicating where in [11] one can find the proofs of the claims.

To study the degeneration one starts with the product $\mathcal{Y}:=Y\times\Delta$, with $\Delta\subset\mathbb{C}$ the unit disc. Before blowing up, the normal bundle of $G\times\{0\}$ in $\mathcal{Y}$ is $N_{G/Y}\oplus\mathcal{O}_G$, where $N_{G/Y}$ is the normal bundle of G in Y.

To blow up, one chooses sections $p_i(t)$, $i=1,\ldots r$, in $\mathcal{Y}$, which are general points of Y for general $t\in\Delta$, and are general points of G when $t=0$. Letting $\mathcal{X}$ be the blowup of $\mathcal{Y}$ along the sections, one checks that the normal bundle of $G_0\times\{0\}$ in $\mathcal{X}$ is obtained by performing r elementary transformations on $N_{G/Y}\oplus\mathcal{O}_G$. Under the numerical conditions above, ($r\geqslant|G\cdot G|$; $(G\cdot G)-r<0$ and even if G has genus zero), a generic choice of degeneration will ensure that these elementary transformations result in a semistable bundle. This result is a combination of [11, Lemma 3.4, Corollary 3.5, and Lemma 3.6].

Choose such a generic degeneration and let $\mathcal{N}$ be the resulting normal bundle of G_0 in $\mathcal{X}$. Since $\deg_{G_0}(\mathcal{N})=\deg(N_{G/Y})-r=G_0\cdot G_0$, $\mathcal{N}$ is a semistable bundle of slope $(G_0\cdot G_0)/2$.

Next, if C is an effective class on the blowup of r general points, we may choose a family of effective curves $\mathcal{C}_t$, $t\in\Delta$, such that each $\mathcal{C}_t\subset\mathcal{X}_t$ has class C. Let $\mathrm{mult}_{G_0}(\mathcal{C}_0)$ denote the multiplicity of G_0 in the limiting curve $\mathcal{C}_0$. Under the condition that $\mathcal{N}$ (and thus also its dual $\mathcal{N}^*$) is semistable, the statement of [11, Corollary 2.2] is that

$$\mathrm{slope}(\mathcal{N}^*)\,\mathrm{mult}_{G_0}(\mathcal{C}_0)\geqslant-(C\cdot G_0).$$

Since $\mathrm{slope}(\mathcal{N}^*)=-(G_0\cdot G_0)/2$, we conclude that $\mathrm{mult}_{G_0}(\mathcal{C}_0)\geqslant 2\left(\frac{C\cdot G_0}{G_0\cdot G_0}\right)$, and thus that $C-2\left(\frac{C\cdot G_0}{G_0\cdot G_0}\right)G_0$ is an effective class on X_0. □

Remarks 8.2. (1) As is clear from the proof, the extra factor of 2 comes from the semistability of $\mathcal{N}^*$ combined with the multiplicity estimate of [11, Corollary 2.2].

(2) Since the multiplicity of G_0 in $\mathcal{C}_0$ is an integer, the estimate $\mathrm{mult}_{G_0}(\mathcal{C}_0)\geqslant 2\left(\frac{C\cdot G_0}{G_0\cdot G_0}\right)$ implies the stronger result that

(a') If C is an effective class on X, then $C-\left\lceil 2\left(\frac{C\cdot G_0}{G_0\cdot G_0}\right)\right\rceil G_0$ is an effective class on X_0.

Here $\lceil\cdot\rceil$ denotes the round-up. We have chosen to record the result without the round-up for several reasons. First, because of the elegance of expressing the result as a reflection, which leads easily to the statement in part (b) of the theorem. Second, if the class of C is sufficiently divisible (e.g., if $C\cdot G_0$ is a multiple of $G_0\cdot G_0$) then the round up makes no difference. However, if one knows more precise information about the class C (enough to determine that $(C\cdot G_0)/(G_0\cdot G_0)$ is not an integer), then the version above may be useful.

(3) If ξ_0 is an integral class, $\varphi_{G_0}(\xi_0)$ may only be a rational class, and then it is natural to scale to make ξ integral. Thus, the fact that $\xi\cdot\xi=\xi_0\cdot\xi_0$ essentially only ensures that, after scaling, $(m\xi)\cdot(m\xi)$ has the same sign (here meaning >0 or $=0$) as $\xi_0\cdot\xi_0$. Since we are mostly interested in reflecting square-zero classes, the scaling makes no difference.

(4) Since the multiplicities of G_0 are symmetric (they are all 1), reflection in G_0 preserves the subspace of symmetric classes. Therefore if ξ_0 is a symmetric nef class on X_0, $\varphi_{G_0}(\xi_0)$ is a symmetric nef class on X.

(5) More generally, one could also obtain symmetric nef classes by finding nef classes on X (for instance by reflection), and restricting them to the subspace $V_r \subset V$ of symmetric classes. Because V_r is self-dual under the intersection product, restriction of the linear form defined by a class ξ amounts to orthogonal projection of ξ onto V_r.

Specifically, if $\xi \in V$ is a nef class on X, then ξ decomposes as $\xi = \xi_r + \xi^\perp$, with $\xi_r \in V_r$, and $\xi^\perp$ in the orthogonal subspace $V_r^\perp$ to V_r. ($V_r^\perp$ consists of those classes of the form $\sum m_i E_i$, with $\sum m_i = 0$). Since the decomposition is orthogonal, we have $\xi^2 = (\xi_r)^2 + (\xi^\perp)^2$. Since the intersection form is negative definite on $V_r^\perp$, if $\xi \notin V_r$ (i.e., if $\xi^\perp \neq 0$) then $\xi_r^2 > \xi^2$.

Thus, if one is searching for nef classes in V_r which are square-zero (classes imposing the strongest conditions, since they are clearly on the boundary of the nef cone), one cannot do it by restricting nef classes to V_r unless they were already in V_r to begin with.

9. Inner square-zero nef classes via reflections

9.1. Construction of the classes

Theorem 9.1. *Let X be the blowup of $\mathbb{P}^1 \times \mathbb{P}^1$ at r general points, $r \geqslant 9$, and let e be an integer such that*

(a) $2 \leqslant e \leqslant \frac{r-4}{2}$ *if r is even, or*
(b) $2 \leqslant e < \frac{r}{4}$ *if r is odd.*

Then the class $\xi(e,r) := (2e^2, r, -2e)$ is an inner nef class on X and satisfies $\xi(e,r)^2 = 0$ (i.e., $\xi(e,r)$ is on the square-zero cone). The same statements hold for the class $(r, 2e^2, -2e)$ obtained by switching the bidegrees.

Proof. Let $(e_1, e_2) = (e, 1)$ if r is even, and $(e_1, e_2) = (e, 2)$ if r is odd. The class $\xi(e,r)$ is obtained, up to multiple of $r - 2e_1e_2$, by reflecting the fibre class $\xi_0 := F_2 = (0,1,0)$ in the curve G_0 obtained by specializing the r general points to lie on a smooth curve G of bidegree (e_1, e_2).

We now check that such a curve G satisfies the numerical conditions of Theorem 8.1. We recall that a smooth curve of bidegree (e_1, e_2) on $\mathbb{P}^1 \times \mathbb{P}^1$ has genus $(e_1 - 1)(e_2 - 1)$.

- If r is even, then $(G \cdot G) - r = 2e - r$ is clearly even, a necessary condition since a curve of bidegree $(e, 1)$ has genus 0. It is also clear that $(G \cdot G) - r < 0$ whenever $e < \frac{r}{2}$, a weaker condition than the condition $e \leqslant \frac{r-4}{2}$ being imposed above in (a).
- If r is odd, then since $e \geqslant 2$, the curve G of bidegree $(e, 2)$ has genus $\geqslant 1$, and thus there is no restriction on the parity of $(G \cdot G) - r$. The only condition needed to apply the theorem is that $(G \cdot G) - r = 4e - r < 0$. This clearly holds whenever $e < \frac{r}{4}$, which is one of the conditions being imposed in (b).

Thus we may apply the theorem. Since the class F_2 is clearly nef on the specialization X_0, part (b) of Theorem 8.1, along with the fact that $r - 2e_1e_2 > 0$ guarantees that $\xi(e,r) = (r - 2e_1e_2)\varphi_{G_0}(\xi_0)$ is nef on X.

It is easy to check directly that $\xi(e,r)$ is square-zero. On the other hand, this also follows from part (b) of Theorem 8.1. Since $\xi_0 = F_2$ is square-zero, so is its reflection $\varphi_{G_0}(\xi_0)$, and therefore so is its scalar multiple $\xi(e,r)$.

We have $\xi(e,r) \cdot K_X = -4e^2 + 2er - 2r = -4\Big((e-1)^2 - \left(\frac{r-4}{2}\right)(e-1) + 1\Big)$. Thus, to have $\xi(e,r) \cdot K_X > 0$ (i.e., in order that $\xi(e,r)$ be an inner class), we need

$$(e-1)^2 - \left(\frac{r-4}{2}\right)(e-1) + 1 < 0.$$

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With the substitution $t = e-1$, the left-hand side of the inequality above becomes the minimal polynomial for α_r and β_r from §1.2. Thus $\xi(e,r) \cdot K_X > 0$ when $e \in (\beta_r + 1, \alpha_r + 1)$.

Since $\beta_r \in (0,1)$, and e is an integer, the condition $\beta_r + 1 < e$ is equivalent to $2 \leqslant e$. If $r \geqslant 9$ then $e < \frac{r}{4} < \frac{r-3}{2} \leqslant \alpha_r + 1$, where the last inequality comes from Lemma 4.6. Thus when r is odd the conditions in (b) imply that $e \in (\beta_r + 1, \alpha_r + 1)$.

If r is even then Lemma 4.6 shows that $\lfloor \alpha_r + 1 \rfloor = \frac{r-4}{2}$, and thus since e is an integer (and α_r is not an integer when $r \geqslant 10$), the condition $e < \alpha_r + 1$ is equivalent to the condition $e \leqslant \frac{r-4}{2}$. Thus when r is even the conditions in (a) are equivalent to the condition that $e \in (\beta_r + 1, \alpha_r + 1)$.

This finishes the proof that the class $\xi(e,r)$ has the properties claimed. By symmetry the class $(r, 2e^2, -2e)$ also has these properties. □

Corollary 9.2. *Suppose that r is even, $r \geqslant 10$, and that e is an integer satisfying $2 \leqslant e \leqslant \frac{r-4}{2}$. Then for all $n \in \mathbb{Z}$ the classes $T_r^n(\xi(e,r))$ are square-zero inner nef classes.*

Proof. Clear since T_r is an automorphism of the problem, preserving nef classes, self-intersections, and the canonical bundle. □

Remarks 9.3. (1) In Corollary 9.2 the orbits of the classes $\xi(e,r)$ under T_r are distinct, but if one also allows scaling by positive integers, then there are fewer equivalence classes.

For instance, when $r = 10$, $\xi(2,10)$ and $\xi(3,10)$ (the only values of e possible in this case) and the classes obtained by switching the bidegrees, are all, up to scaling, in the same orbit. Similarly, when $r = 12$ the classes $\xi(2,12)$, $\xi(3,12)$, and $\xi(4,12)$ and the classes obtained by switching the bidegrees are, up to scaling, in the same orbit. When $r = 14$ there are two equivalence classes, those for which the minimal integral ray generator on the ray spanned by $\xi(e,14)$ has intersection 6 with K_X ($e = 2, 5$), and those for which the intersection is 10 ($e = 3, 4$).

(2) The conclusion of Theorem 9.1 also holds when $r = 8$. Then we must have $e = 2$, and $\xi(2,8)$ is the class $(8,8,-4)$, a multiple of $(2,2,-1)$, which is also a multiple of $v_{\alpha_8} = v_{\beta_8}$. This class is inner, but not strictly inner (α_r and β_r are rational numbers only when $r = 8$ or $r = 9$, so in the even case $r = 8$ is only case a rational class could have slope α_r or β_r). See also §5.2.

(3) Here is an explanation of the curve classes used in Theorem 9.1. Let G have bidegree (e_1, e_2). In order to apply Theorem 8.1 we need to have $(G \cdot G) - r = 2e_1e_2 - r < 0$. This puts restrictions on the sizes of e_1 and e_2.

The one class which is clearly nef on the specialization X_0 is the fibre class F_2 (or symmetrically, F_1). With G_0 the curve obtained as the proper transform of G when the points are specialized onto G, we have

$$\varphi_{G_0}(F_2) = \frac{1}{(r - 2e_1e_2)}\left(2e_1^2,\ r,\ -2e_1\right).$$

Surprisingly (after scaling), the result only depends on e_1. Thus, in light of the condition $2e_1e_2 < r$, in order to obtain as wide a range of reflections as possible, we should make e_2 as small as possible.

When r is even, we may take $e_2 = 1$, since $(G \cdot G) - r$ will always be even. When r is odd, we must take $e_2 \geqslant 2$ (and $e_1 \geqslant 2$) in order to ensure that G has genus $\geqslant 1$. From these choices the rest of the conditions in Theorem 9.1(a) follow directly from the further requirement that the reflection of F_2 be an inner class.

(4) When r is even, if one takes $e = \frac{r-2}{2}$, then by the arguments in the proof of Theorem 9.1, the class $\xi(e,r)$ is a nef *outer* class on the square-zero cone. These have all been determined in Theorem 3.3, and all are positive multiples of the classes $T_r^n(F_2)$, $n \in \mathbb{Z}$. In this case one can check that $\xi\left(\frac{r-2}{2}, r\right) = 2 \cdot T_r^{-2}(F_2)$.

9.2. On the possibility of reflecting other classes

As remark (3) above explains, Theorem 9.1 is based on the fact that the class F_2 is guaranteed to be nef on X_0. When r is even, and the points in general position we know that the classes $T_r^n(F_2)$, $n \in \mathbb{Z}$, are nef classes on X, and on the square-zero cone. If we knew that some or all of these classes remain nef after we specialize the points (i.e., on X_0), then we could reflect those classes, and obtain other square-zero inner nef classes. We conclude this section with some particularly interesting examples of such potential calculations.

In the case $r = 10$, let us consider specializing the points onto a curve G of bidegree $(3, 1)$. In this case, we know that the classes $T_{10}^n(F_2)$ are not nef on X_0 when $n \leqslant -2$, since we can compute that those particular classes intersect negatively with G_0.

We (the authors) do not know if the classes $T_{10}^n(F_2)$ remain nef on X_0 when $n \geqslant 1$. Suppose for a moment that they do. Then, by continuity, the limiting class $v_{\alpha_{10}}$ would also be nef on X_0, and therefore its reflection $\varphi_{G_0}(v_{\alpha_{10}})$ would be a square-zero inner nef class on X. However, up to multiple, $\varphi_{G_0}(v_{\alpha_{10}})$ is the class $(1, 1, -\frac{1}{\sqrt{5}})$, and thus we would have shown the existence of an irrational Seshadri constant, something suspected, but not yet known to exist.

That is, the argument above is

- If the classes $T_{10}^r(F_2)$ are nef on X_0 for all $n \geqslant 1$ (or all sufficiently large n), then $v_{\alpha_{10}}$ is nef on X_0.
- If $v_{\alpha_{10}}$ is nef on X_0 then the class $(1, 1, -\frac{1}{\sqrt{5}})$ is nef when the points are in general position, and therefore irrational Seshadri constants exist.

We do not know if this observation is a step forward in producing an irrational Seshadri constant, or just another way of hiding the crucial issue.

The surface X_0 can also be realized as the blowup of the Hirzebruch surface $\mathbb{F}_4$ at 10 general points. (The curve of self-intersection (-4) is G_0, and the class F_1 is the fibre class of the morphism $\mathbb{F}_4 \longrightarrow \mathbb{P}^1$.) For the record we include the change of basis matrix on the symmetrized Néron-Severi lattice for these two realizations of the surface.

$$M = \begin{array}{c} \\ B \\ F' \\ E' \end{array}\!\!\begin{array}{c} \begin{array}{ccc} F_1 & F_2 & E \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 1 & 7 & 10 \\ 0 & -1 & -1 \end{bmatrix} \end{array}$$

Here B is the class of the curve of self-intersection -4, F' is the fibre class, and E' the sum of the exceptional divisors.

Thus, the question above is whether, in the basis coming from $\mathbb{F}_4$, the classes $M \cdot T_{10}^n(F_2)$, $n \geqslant 1$, are nef on the blowup of $\mathbb{F}_4$ at 10 general points. For instance, when $n = 1$ this is asking if the class $5B + 26F' - 4E'$ is nef.

More generally, for even $r \geqslant 10$, if one knew that that the class v_{α_r} was nef on the surface X_0 which resulted from specializing the points to lie on a curve of bidegree $(\frac{r-4}{2}, 1)$ (the last possible case in (a) of Theorem 9.1), then one would know that the reflection $\varphi_{G_0}(v_{\alpha_r})$ was nef when the points were in general position. This reflection is, up to multiple, the class $(\frac{r-8}{2}, 1, -\sqrt{\frac{r-8}{r}})$, and would again provide an example of an irrational Seshadri constant.

This surface X_0 can again be realized as the blowup of the Hirzebruch surface $\mathbb{F}_4$ at r general points (the curve G_0 again has self-intersection -4.) Thus, it seems very interesting to investigate nef classes on the blowup of $\mathbb{F}_4$ at r general points, $r \geqslant 10$, r even. The classes in question all lie in the K_X-negative part of the effective cone. However, a curve which shows that such a class is not nef must be K_X-null or K_X-positive, which is where the difficulty of the question lies.

Here is a potentially useful restatement of the above question. As the proof of Theorem 3.3 (implicitly) shows, each class $\xi_{n,r} = T_r^n(F_2)$ is a convex combination of $C_{n,r}$ and $C_{n+1,r}$, where $C_{n,r} = T_r^n(E)$. Specifically, since $E + T_r(E) = (0, 0, 1) + (0, r, -1) = (0, r, 0) = r\xi_{0,r}$, it follows that

$$\xi_{n,r} = \frac{1}{r}\left(C_{n,r} + C_{n+1,r}\right) \quad \text{for all } n \in \mathbb{Z}. \tag{9.1}$$

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The class E is the disjoint union of the r exceptional divisors, and so each class $C_{n,r}$ is the disjoint union of $r(-1)$-curves. If these components of $C_{n,r}$ and $C_{n+1,r}$ remain irreducible when specializing the points to general points of a curve of bidegree $\left(\frac{r-4}{2}, 1\right)$, then it would follow that $\xi_{n,r}$ is a nef class on the specialization. Thus, the more general question above can be rephrased as :

Q : For some even r, $r \geqslant 10$, is it true that for sufficiently large n, the (-1)-curves which are the components of $C_{n,r}$ remain irreducible when the r points are specialized to general points of a curve of bidegree $(\frac{r-4}{2}, 1)$?

One can also rephrase this even more explicitly. Since the specialization is isomorphic to the blowup of $\mathbb{F}_4$ at r general points, we can write the (-1)-classes in the natural basis from this point of view. The question then becomes "are these (-1)-classes irreducible for all sufficiently large n?".

As discussed above, a positive answer, for any fixed even $r \geqslant 10$, would establish the existence of an irrational Seshadri constant.

10. Inner square-zero nef classes via pullbacks

In this section we use pullback maps to produce inner square-zero nef classes. When r has a factor r_0 which is even and $\geqslant 8$, this allows us to produce such classes different from the classes in §9.

Given positive integers a, b, let $\varphi_{a,b} \colon \mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ be a map which is of degree a on the first $\mathbb{P}^1$ factor, and degree b on the second. For instance, $([x_0 : x_1], [y_0 : y_1]) \mapsto ([x_0^a : x_1^a], [y_0^b : y_1^b])$ is such a map. The map $\varphi_{a,b}$ has degree ab.

Fix a positive integer r_0, and let X_{r_0} be the blowup of $Y = \mathbb{P}^1 \times \mathbb{P}^1$ at r_0 general points, $p_1, \ldots, p_{r_0}$. Since the points are general we may assume that they do not lie in the branch locus of φ_{ab}, and so each of the points p_i pulls back to ab distinct points.

Let X_r be the blowup of $\mathbb{P}^1 \times \mathbb{P}^1$ at the resulting $r := abr_0$ points, and

$$\psi_{a,b} \colon X_r \longrightarrow X_{r_0}$$

the induced morphism.

As in the rest of the paper, we use V_r and V_{r_0} respectively for the subspaces of $H^2(X_r, \mathbb{R})$ and $H^2(X_{r_0}, \mathbb{R})$ generated by the fibre classes and the sum of the exceptional divisors. The morphism $\psi_{a,b}$ induces pullback and pushforward morphisms between these spaces. Specifically,

$$\begin{array}{rcccl} & V_r & \longleftarrow & V_{r_0} & : \psi^*_{a,b} \\ & (ad_1, bd_2, -m) & \longleftarrow\!\!\shortmid & (d_1, d_2, -m) & \\ \hline \psi_{a,b*}\colon & V_r & \longrightarrow & V_{r_0} & \\ & (d_1', d_2', -m') & \longmapsto & (bd_1', ad_2', -abm') & \end{array} \tag{10.1}$$

Here, as before, the coordinates on V_r and V_{r_0} are with respect to the basis consisting of the fibre classes and the sum of the exceptional divisors.

The pullback and pushforward morphisms are adjoint with respect to the inner products on the two spaces. For $v = (d_1, d_2, -m) \in V_{r_0}$ and $w = (d_1', d_2', -m') \in V_r$,

$$\left\langle \varphi^*_{a,b}(v),\, w \right\rangle_r = ad_1d_2' + bd_2d_1' - abr_0mm' = \left\langle v,\, \varphi_{a,b*}(w) \right\rangle_{r_0}. \tag{10.2}$$

In the equation above we have used $\langle \bullet, \bullet \rangle$ instead of $\cdot$ for the inner product, to allow us to write a subscript indicating on which space the inner product is being evaluated.

For classes $v_1, v_2 \in V_{r_0}$, and $w_1, w_2 \in V_r$, we also have

$$\left\langle \psi^*_{a,b}(v_1),\, \psi^*_{a,b}(v_2) \right\rangle_r = ab\langle v_1,\, v_2\rangle_{r_0} \quad \text{and} \quad \left\langle \psi_{a,b*}(w_1),\, \psi_{a,b*}(w_2) \right\rangle_{r_0} = ab\langle w_1,\, w_2\rangle_r. \tag{10.3}$$

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The r points at which we are blowing up are not general. However, effective classes remain effective under specialization, and dually, classes which are nef when the points are specialized are nef when the points are in general position. Thus, if ξ_0 is a nef class in V_{r_0}, $\psi^*_{a,b}(\xi_0)$ is a nef class on $\mathbb{P}^1 \times \mathbb{P}^1$ blown up at r general points. Here we are identifying the intersection spaces for the blowup at general points, and the blowup at special points as in Theorem 8.1.

10.1. Construction of the classes

Proposition 10.1. *Let $\xi_0 = (d_1, d_2, -m) \in V_{r_0}$ be a point in the octant where d_1, d_2, and m are $\geqslant 0$. Fix positive integers a and b, and set $\xi = \psi^*_{a,b}(\xi_0) \in V_r$.*

(a) *If $\xi_0^2 = 0$ then $\xi^2 = 0$.*
(b) *If ξ_0 is nef, then ξ is a nef class on $\mathbb{P}^1 \times \mathbb{P}^1$ blown up at r general points.*
(c) *If ξ_0 is $K_{X_{r_0}}$ positive, then ξ is K_{X_r} positive.*

That is, if ξ_0 is square zero, nef, or K_X-positive, the same is true of the pullback ξ. In particular, inner square-zero nef classes on X_{r_0} pull back to inner square-zero nef classes on $\mathbb{P}^1 \times \mathbb{P}^1$ blown up at r general points.

Proof. (a) By (10.3) we have $\langle \xi, \xi \rangle_r = ab\langle \xi_0, \xi_0 \rangle_{r_0} = ab\,\xi_0^2 = 0$, so ξ is a square-zero class. (b) The class ξ is nef by the argument above: when the r points are specialized, the class ξ is nef on the specialization, and hence nef when the points are in general position.

(c) To see that $\left\langle K_{X_r}, \xi \right\rangle_r > 0$, we use (10.2), and show that $\left\langle \psi_{a,b*}(K_{X_r}),\ \xi_0 \right\rangle_{r_0} > 0$. Since $K_{X_r} = (-2, -2, 1)$, and assuming by symmetry that $a \leqslant b$, we have

$$
\begin{aligned}
\psi_{a,b*}(K_{X_r}) &\overset{(10.1)}{=\!=} (-2b, -2a, ab) \\
&= b\,(-2, -2, 1) + (b-a)\,(0, 2, 0) + (a-1)b\,(0, 0, 1) \\
&= b\,K_{X_{r_0}} + 2(b-a)\,F_2 + (a-1)b\,E,
\end{aligned}
$$

all the coefficients above are $\geqslant 0$ and $b > 0$. By assumption $\left\langle K_{X_{r_0}}, \xi_0 \right\rangle_{r_0} > 0$. Furthermore $\langle F_2, \xi_0 \rangle = d_1$ and $\langle E, \xi_0 \rangle = m$, both of which are $\geqslant 0$ by assumption on the octant. Thus $\left\langle K_{X_r}, \xi \right\rangle_r > 0$. □

Remarks 10.2. (1) As clear from the proof of (c), even a class ξ_0 such that $\langle K_{X_{r_0}}, \xi_0 \rangle_{r_0} < 0$ can pull back to a class ξ with $\langle K_{X_r}, \xi \rangle_r > 0$, as long as the intersections of ξ_0 with $2(b-a)F_2$ and $(a-1)bE$ are sufficiently positive to make up for the negativity of the first intersection.

In particular as long as $(a, b) \neq (1, 1)$ (i.e., as long as $\psi_{a,b}$ is not the identity map), the $K_{X_{r_0}}$-null classes $v_{\alpha_{r_0}}$ and $v_{\beta_{r_0}}$ pull back to classes which are K_{X_r} positive.

(2) We also note that if ξ_0 is nef, ξ_0 must be in the octant with d_1, d_2, and $m \geqslant 0$ (§2.1).

Let a, b, r_0, and $r = abr_0$ be as above. Starting with a class $\xi(e_0, r_0) = (2e_0^2, r_0, -2e_0)$ produced by Theorem 9.1, we compute that

$$
\psi^*_{a,b}(\xi(e_0, r_0)) = (2ae_0^2, br_0, -2e_0) = \tfrac{1}{a}(2(ae_0)^2, abr_0, -2ae_0) = \tfrac{1}{a}\xi(e, r)
$$

where $e = ae_0$. From the fact that e_0 satisfies the inequalities necessary to apply Theorem 9.1 (i.e., that $2 \leqslant e_0 \leqslant \frac{r_0-4}{2}$ or $2 \leqslant e_0 < \frac{r_0}{4}$ depending on the parity of r_0), we see that e also satisfies the corresponding inequalities.

That is, pulling back the classes as produced by Theorem 9.1 only gives classes of the same type, up to scalar multiple.

However, if r_0 is even, $r_0 \geqslant 8$, then as recorded in Corollary 9.2, we may apply powers of T_{r_0} to each $\xi(e_0, r_0)$ to obtain infinitely many other inner nef square-zero classes. We may then pull back these classes to X_r and apply powers of T_r. In general the pullback of points in the T_{r_0}-orbit of $\xi(e_0, r_0)$ are not in the T_r orbit of the pullback of any $\xi(e_0', r_0)$, and this allows us to produce infinitely many new inner nef square-zero classes.

To illustrate the idea, we look at one of the smallest possible examples.

Example 10.3. Let $r_0 = 10$, and $(a, b) = (2, 1)$, so that $r = abr_0 = 20$. Taking $e_0 = 2, 3$ in Theorem 9.1(a) the classes $\frac{1}{2}\xi(2, 10) = (4, 5, -2)$ and $\frac{1}{2}\xi(3, 10) = (9, 5, -3)$ are square-zero inner nef classes, as are the classes $(5, 4, -2)$ and $(5, 9, -3)$ obtained by switching the first two coordinates. Here we have divided by 2 to remove common factors among the coordinates, that is, to replace each class by the integral generator of the ray it spans.

Let us just focus on one of these, the class $\xi_0 := \frac{1}{2}\xi(2, 10) = (4, 5, -2)$.

The forward orbit $T_{10}^n(\xi_0)$, $n \geqslant 1$, of ξ_0 converges, up to scaling, to $v_{\alpha_{10}}$. As noted in §10.2, $v_{\alpha_{10}}$ will pull back to a K_X-positive class (and also a nef, square zero class, by Proposition 10.1). The sequence $\xi_{n,10} = T_{10}^n(F_2)$, $n \geqslant 1$ also converges, modulo scaling, to $v_{\alpha_{10}}$, although from the "other side". It follows that for n sufficiently large, the $\xi_{n,10}$ also pull back to K_X-positive classes.

Thus, by pulling back, we obtain infinitely many inner nef square-zero classes converging (modulo scaling) to $\psi_{2,1}^*(v_{\alpha_{10}})$, and converging from both sides. The example is illustrated in Figure 16.

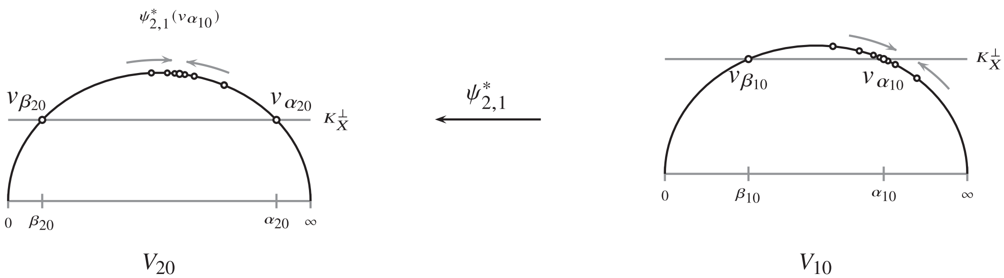

Figure 16. *Illustration of the pullback map.*

Now we can apply T_{20}^m to the pullbacks. Applying T_{20}^m to $\psi_{2,1}^*(v_{\alpha_{10}})$ we obtain a sequence of inner, nef, square-zero classes converging (modulo scaling) to $v_{\beta_{20}}$ or $v_{\alpha_{20}}$ as $m \to -\infty$ or $m \to \infty$ respectively.

But, each of the elements in this sequence itself has a sequence of inner, nef, square-zero classes converging to it, from both sides. Specifically, fixing m, the sequences

$$T_{20}^m(\psi_{2,1}^*(T_{10}^n(\xi_0))) \quad \text{and} \quad T_{20}^m(\psi_{2,1}^*(\xi_{n,10}))$$

converge (modulo scaling) to $T_{20}^m(\psi_{2,1}^*(v_{\alpha_{10}}))$ as $n \to \infty$. For all n the classes of the first type are inner, nef, and square-zero classes. Classes of the second type are nef and square-zero, and are inner (i.e., K_X-positive) for sufficiently large n. In this specific example, $n \geqslant 1$ is large enough.

We can also apply this construction to the three other classes (e.g., $(9, 5, -3)$) listed above.

By repeated pullbacks we can thus arrive at an r where we can find an infinite sequence of inner, nef, square zero classes, each member of which has an infinite sequence of such classes converging to it, and each member of those previous sequences has an infinite sequence of such classes converging to it, . . . , and so on, up to a finite number of such steps.

As discussed in §1.8, a consequence of the SHGH conjecture is that some portion of the nef cone should be round. If the nef cone is not round then the above examples suggest that the actual description of the nef cone is likely to be quite complicated.

10.2. Use in lower bounds for the Seshadri constant

The paper [5] establishes lower bounds on the Seshadri constants for line bundles whose Seshadri constants are not affected by (-1)-curves. As noted in §1.10, for a line bundle L of type (e_1, e_2), these bounds are

$$\begin{aligned}\epsilon_r(L) &\geqslant \eta_r(L)\Big(1-\tfrac{1}{5r}\Big)^{\frac{1}{2}} \text{ for } r \text{ odd}, \tfrac{e_2}{e_1} \in \big[\tfrac{2}{(\sqrt{r}-1)^2}, \tfrac{(\sqrt{r}-1)^2}{2}\big]\\ \epsilon_r(L) &\geqslant \eta_r(L)\Big(1-\tfrac{2}{9r}\Big)^{\frac{1}{2}} \text{ for } r \text{ even}, \tfrac{e_2}{e_1} \in [\beta_r, \alpha_r].\end{aligned} \tag{10.4}$$

In §9 we have constructed examples of inner bundles where $\epsilon_r(L) = \eta_r(L)$ (all of the classes $\xi(e, r)$, and, when r is even, their orbits under T_r). Applying pullbacks we can construct even more such classes when r has an even factor $\geqslant 8$. Thus, the methods of these sections produce examples of bundles whose Seshadri constants are larger than the lower bounds above.

The convex hull of such classes (for fixed r) then also provides a lower bound on the Seshadri constant. This lower bound is exact for bundles of the type we have constructed (those where $\epsilon_r(L) = \nu_r(L)$), and improves on the bounds in (10.4), at least in the neighbourhood of such bundles.

The authors have been unable to find a useful way to describe and organize all the bundles produced by these procedures, and thus are unable to give a short formula for a better lower bound. Thus, the bounds in (10.4) seem, at the moment, to be the most generally useful. They are also quite strong. For instance, when $r = 20$, the lower bound in (10.4) is that $\epsilon_r(L)$ is differs from $\eta_r(L)$ by a factor of no worse than $\sqrt{\frac{89}{90}} \approx 0.994428\ldots$.

We end this section by giving an application of the formulas in §10 to establish a reasonably strong family of bounds on the symmetric effective cone, valid (at least) whenever $8 \mid r$.

Theorem 10.4. *Suppose that r_0 and $\xi_0 \in V_{r_0}$ are such that for all effective classes $C \in V_{r_0}$ one has*

$$-r_0 C^2 \leqslant (C \cdot \xi_0)^2. \tag{10.5}$$

Then

(a) *For any positive a, b, setting $r = abr_0$ and $\xi = \psi^*_{a,b}(\xi_0)$, for all effective classes $C \in V_r$ we similarly have*

$$-rC^2 \leqslant (C \cdot \xi)^2. \tag{10.6}$$

(b) *If r_0 is even, then for each $n \in \mathbb{Z}$ (10.6) holds with $r = r_0$, $\xi = T^n_{r_0}(\xi_0)$, and for all effective classes $C \in V_{r_0}$.*

(c) *Condition (10.5) of the theorem holds when $r_0 = 8$ and $\xi_0 = -K_{X_8}$.*

Here the condition on effectivity means “for $\mathbb{P}^1 \times \mathbb{P}^1$ blown up at r_0 (respectively r) general points”.

Proof. (a) Let C be an effective class in V_r. Since C is a class which is effective on $\mathbb{P}^1 \times \mathbb{P}^1$ blown up at r general points, then it is also an effective class when blowing up at special points. Thus, $\psi_{a,b*}(C)$ is also an effective class in V_{r_0}.

We then have

$$-r\langle C, C\rangle_r \overset{(10.3)}{=\!=} -r_0\langle \psi_{a,b*}(C), \psi_{a,b*}(C)\rangle_{r_0} \overset{(10.5)}{\leqslant} \langle \psi_{a,b*}(C), (-K_{X_8})\rangle^2_{r_0} \overset{(10.2)}{=\!=} \langle C, \xi\rangle^2_r,$$

or $-rC^2 \leqslant (C \cdot \xi)^2$, which was the inequality to be proved in this case.

(b) This follows from (10.6) and the fact that, by Theorem 3.1, T_{r_0} preserves the intersection form and the property of being effective.

(c) Let X_8 denote $\mathbb{P}^1 \times \mathbb{P}^1$ blown up at 8 general points. For the curve classes $C_n = (4n(n-1), 4n(n+1), 1-2n^2)$ of §5.2, we compute that $C_n^2 = -8$ and $C_n \cdot (-K_{X_8}) = 8$. Thus these curves, which, by the arguments in §5.2 form the boundary of the effective cone, satisfy $-8(C_n^2) = \big(C_n \cdot (-K_{X_8})\big)^2$. By convexity, it follows that for all effective classes $C \in V_8$ we have $-8C^2 \leqslant \big(C \cdot (-K_{X_8})\big)^2$. □

Remarks 10.5. (1) The bound (10.6) is homogeneous in C, but not in ξ. (2) We recall that $-K_{X_8}$ is a square-zero nef class. Thus by Proposition 10.1 and Theorem 3.1 each class $\xi = T^n_r(\psi^*_{a,b}(-K_{X_8}))$ is also a square-zero nef class, and is an inner class as long as $ab \neq 1$ (i.e., $r \neq 8$).

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(3) The form of the theorem is set up to be able to iterate the process. For instance, if $8 \mid r$ and if $\frac{r}{8}$ has many factors, one can choose different combinations of (a) and (b) to step from $r_0 = 8$ to r.

(4) Here is an illustration of how this bound works. In the diagram below, in V_8, one can see the class $\xi_0 = -K_X$, the corresponding line $\xi_0^\perp$ (tangent to the square-zero cone), and the dashed curve, also tangent to the square-zero cone at ξ_0, which contains the boundary effective classes when $r_0 = 8$.

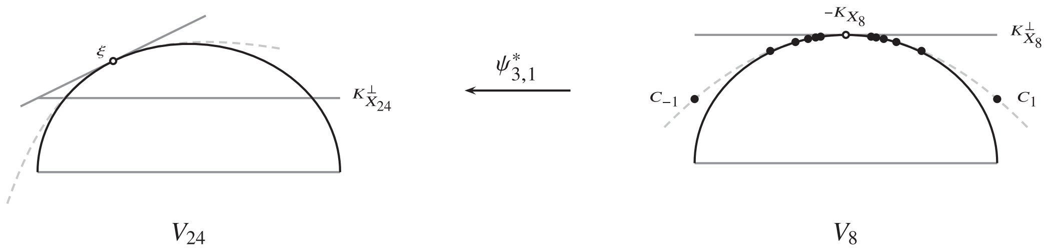

Figure 17. *Illustration of the bound.*

Pulling this back via $\psi_{3,1}$, we obtain a class ξ in V_{24}, which is an inner square-zero nef class. The dashed curve pulls back to a curve (the curve $24C^2 = (C \cdot \xi)^2$) which bounds all effective classes. That curve is tangent to the square-zero cone at ξ, and in a neighbourhood of ξ stays very close to the cone.

The curve does a worse job of bounding the effective classes farther away from ξ. But, by applying powers of T_{24} we can shift ξ, and the bounding curve, and obtain a family of bounds, which together tightly restrict the possible K_X-positive effective curve classes.

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