POSITIVE SYSTEMS OF KOSTANT ROOTS

IVAN DIMITROV AND MIKE ROTH

ABSTRACT. Let $\mathfrak{g}$ be a simple complex Lie algebra and let $\mathfrak{t} \subset \mathfrak{g}$ be a toral subalgebra of $\mathfrak{g}$. As a $\mathfrak{t}$-module $\mathfrak{g}$ decomposes as

$$\mathfrak{g} = \mathfrak{s} \oplus \left(\oplus_{\nu \in \mathcal{R}} \mathfrak{g}^{\nu} \right)$$

where $\mathfrak{s} \subset \mathfrak{g}$ is the reductive part of a parabolic subalgebra of $\mathfrak{g}$ and $\mathcal{R}$ is the Kostant root system associated to $\mathfrak{t}$. When $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}$ the decomposition above is nothing but the root decomposition of $\mathfrak{g}$ with respect to $\mathfrak{t}$; in general the properties of $\mathcal{R}$ resemble the properties of usual root systems. In this note we study the following problem: "Given a subset $\mathcal{S} \subset \mathcal{R}$, is there a parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ containing $\mathcal{M} = \oplus_{\nu \in \mathcal{S}} \mathfrak{g}^{\nu}$ and whose reductive part equals $\mathfrak{s}$?". Our main results is that, for a classical simple Lie algebra $\mathfrak{g}$ and a saturated $\mathcal{S} \subset \mathcal{R}$, the condition $(\mathrm{Sym}^{\cdot}(\mathcal{M}))^{\mathfrak{s}} = \mathbf{C}$ is necessary and sufficient for the existence of such a $\mathfrak{p}$. In contrast, we show that this statement is no longer true for the exceptional Lie algebras $\mathrm{F}_4, \mathrm{E}_6, \mathrm{E}_7$, and E_8. Finally, we discuss the problem in the case when $\mathcal{S}$ is not saturated.

Keywords: Parabolic subalgebras, Kostant root systems, Positive roots.

1. INTRODUCTION

1.1. Let $\mathfrak{g}$ be a simple complex Lie algebra and let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra. The root decomposition of $\mathfrak{g}$ with respect to $\mathfrak{h}$ is

$$\mathfrak{g} = \mathfrak{h} \oplus \left(\oplus_{\alpha \in \Delta} \mathfrak{g}^{\alpha} \right) \tag{1.1}$$

where, for any $\alpha \in \mathfrak{h}^*$,

$$\mathfrak{g}^{\alpha} := \{x \in \mathfrak{g} \mid [t, x] = \alpha(t)x \text{ for every } t \in \mathfrak{h}\} \qquad \text{and} \qquad \Delta = \{\alpha \in \mathfrak{h}^* \setminus \{0\} \mid \mathfrak{g}^{\alpha} \neq 0\}.$$

The Borel subalgebras of $\mathfrak{g}$ containing $\mathfrak{h}$ are in a bijection with the *positive systems* $\Delta^+ \subset \Delta$, i.e., the subsets Δ^+ satisfying the following properties: (i) $\Delta = \Delta^+ \cup (-\Delta^+)$, (ii) $\Delta^+ \cap (-\Delta^+) = \emptyset$, and (iii) $\alpha, \beta \in \Delta^+$, $\alpha + \beta \in \Delta$ implies $\alpha + \beta \in \Delta^+$. Positive systems of roots represent a much studied and well-understood topic in the theory of semisimple Lie algebras. Here is a particular problem that arises in various situations: "Given a subset $\Phi \subset \Delta$, determine if there is a positive system Δ^+ containing Φ". The answer is that such a positive system exists if and only if the semigroup generated by Φ does not contain 0. The aim of this paper is to address the analogous problem in a more general situation.

1.2. Let $\mathfrak{t} \subset \mathfrak{g}$ be a toral subalgebra of $\mathfrak{g}$, that is, a commutative subalgebra of semisimple elements. As a $\mathfrak{t}$-module $\mathfrak{g}$ decomposes as

2010 *Mathematics Subject Classification.* Primary 17B22; Secondary 17B20, 17B25.
Research of I. Dimitrov and M. Roth was partially supported by NSERC grants.

(1.2) $$\mathfrak{g} = \mathfrak{s} \oplus \left(\oplus_{\nu \in \mathcal{R}} \mathfrak{g}^\nu \right)$$

where

$$\mathfrak{g}^\nu := \{x \in \mathfrak{g} \,|\, [t,x] = \nu(t)x \text{ for every } t \in \mathfrak{t}\}, \quad \mathfrak{s} = \mathfrak{g}^0, \quad \text{and} \quad \mathcal{R} = \{\nu \in \mathfrak{t}^* \setminus \{0\} \,|\, \mathfrak{g}^\nu \neq 0\}.$$

We refer to $\mathcal{R}$ as the $\mathfrak{t}$-root system of $\mathfrak{g}$, to the elements of $\mathcal{R}$ as the $\mathfrak{t}$-roots, and to the spaces $\mathfrak{g}^\nu$ as the $\mathfrak{t}$-root spaces. Often we will drop the reference to $\mathfrak{t}$ when it is clear from the context.

To explain the relation between the decompositions (1.1) and (1.2), extend $\mathfrak{t}$ to a Cartan subalgebra $\mathfrak{h}$. The inclusion $\mathfrak{t} \subset \mathfrak{h}$ then induces a surjection $\mathfrak{h}^* \to \mathfrak{t}^*$. The $\mathfrak{t}$-root system $\mathcal{R}$ consists of the nonzero elements of the image of Δ under this map, and for any $\nu \in \mathcal{R}$ the $\mathfrak{t}$-root space $\mathfrak{g}^\nu$ is the sum of the $\mathfrak{h}$-root spaces $\mathfrak{g}^\alpha$ such that $\alpha \mapsto \nu$. Since $\mathfrak{t}$ may be an arbitrary complex subspace of $\mathfrak{h}$ we see that, in contrast to the case of an $\mathfrak{h}$-decomposition, $\mathfrak{t}$-root spaces may be more than one-dimensional, and $\mathfrak{t}$-roots may be complex multiples of one another. (For $\mathfrak{h}$-root systems, $\alpha, r\alpha \in \Delta$ implies that $r = \pm 1$.)

1.3. The subalgebra $\mathfrak{s}$ is a reductive subalgebra of $\mathfrak{g}$ and, moreover, $\mathfrak{s}$ is a reductive part of a parabolic subalgebra of $\mathfrak{g}$. Note that $\mathfrak{t}$ is contained in $\mathcal{Z}(\mathfrak{s})$, the centre of $\mathfrak{s}$. In the case when $\mathfrak{t} = \mathcal{Z}(\mathfrak{s})$ the properties of $\mathcal{R}$ and the decomposition (1.2) were studied by Kostant, [K]. Kostant proved that, for every $\nu \in \mathcal{R}$, $\mathfrak{g}^\nu$ is an irreducible $\mathfrak{s}$-module and showed that $\mathcal{R}$ inherits many of the properties of Δ. To recognize Kostant's contribution, we refer to the elements of $\mathcal{R}$ as "Kostant roots" in the title, however we use the shorter "$\mathfrak{t}$-roots" in the text.

1.4. To describe and motivate the problem we address in this note, we assume in this subsection that $\mathfrak{t} = \mathcal{Z}(\mathfrak{s})$. We caution the reader that not all of equivalences in the following discussion hold when $\mathfrak{t} \neq \mathcal{Z}(\mathfrak{s})$.

One introduces the notion of a positive system $\mathcal{R}^+ \subset \mathcal{R}$ exactly as above: (i) $\mathcal{R} = \mathcal{R}^+ \cup (-\mathcal{R}^+)$, (ii) $\mathcal{R}^+ \cap (-\mathcal{R}^+) = \emptyset$, and (iii) $\mu, \nu \in \mathcal{R}^+$, $\mu + \nu \in \mathcal{R}$ implies $\mu + \nu \in \mathcal{R}^+$. Proposition VI.1.7.20 in [B] implies that positive systems in $\mathcal{R}$ are in a bijection with parabolic subalgebras of $\mathfrak{g}$ whose reductive part is $\mathfrak{s}$. The paper [DFG] contains a detailed discussion (in slightly different terms) of positive systems $\mathcal{R}^+$. In particular, a result of [DFG] implies that a subset $\mathcal{T} \subset \mathcal{R}$ is a positive system if and only if there exists a linear function $\varphi : \mathrm{V} \to \mathbf{R}$, V being the real vector space spanned by $\mathcal{R}$, such that $\ker \varphi \cap \mathcal{T} = \emptyset$ and $\nu \in \mathcal{T}$ if and only if $\varphi(\nu) > 0$. Note that every positive system $\mathcal{R}^+$ is *saturated*, i.e., $\nu \in \mathcal{R}^+, r \in \mathbf{Q}_+$ and $r\nu \in \mathcal{R}$ imply $r\nu \in \mathcal{R}^+$.

In a previous paper [DR] we came across the analogue of the problem mentioned above: "Given a subset $\mathcal{S} \subset \mathcal{R}$ determine whether there is a positive system $\mathcal{R}^+$ containing $\mathcal{S}$". An obvious necessary and sufficient condition (equivalent to the existence of the linear function φ above) for the existence of a positive system $\mathcal{R}^+$ containing $\mathcal{S}$ is the requirement that the semigroup generated by $\mathcal{S}$ does not contain 0. Unfortunately, this combinatorial condition is not easy to verify. On the other hand, in our intended application in [DR], the condition $(\mathrm{Sym}^\cdot(\mathcal{M}))^\mathfrak{s} = \mathbf{C}$ where $\mathcal{M} = \oplus_{\nu \in \mathcal{S}} \mathfrak{g}^\nu$, arose naturally in the context of Geometric Invariant Theory. This latter condition is necessary for the existence of a positive system $\mathcal{R}^+$ as above. To see this, note that $(\mathrm{Sym}^\cdot(\mathcal{M}))^\mathfrak{s}$ always contains at least the constants $\mathbf{C}$, the inclusion $\mathfrak{t} \subset \mathfrak{s}$ implies $(\mathrm{Sym}^\cdot(\mathcal{M}))^\mathfrak{s} \subset (\mathrm{Sym}^\cdot(\mathcal{M}))^\mathfrak{t}$, and the condition that the semigroup generated by $\mathcal{S}$ does not contain 0 is equivalent to $(\mathrm{Sym}^\cdot(\mathcal{M}))^\mathfrak{t} = \mathbf{C}$.

In fact, there is a stronger necessary condition for $\mathcal{S}$ to be contained in a positive system. Since $\mathcal{R}^+$ is saturated, if $\mathcal{S} \subset \mathcal{R}^+$ then $\overline{\mathcal{S}} \subset \mathcal{R}^+$, where $\overline{\mathcal{S}}$ denotes the saturation of S, i.e., $\overline{\mathcal{S}} = \mathbf{Q}_+\mathcal{S} \cap \mathcal{R}$. Set $\overline{\mathcal{M}} := \oplus_{\nu\in\overline{\mathcal{S}}} \mathfrak{g}^\nu$. It is easy to see that $(\mathrm{Sym}^{\cdot}(\mathcal{M}))^{\mathfrak{t}} = \mathbf{C}$ if and only if $(\mathrm{Sym}^{\cdot}(\overline{\mathcal{M}}))^{\mathfrak{t}} = \mathbf{C}$ and that we have the inclusions $(\mathrm{Sym}^{\cdot}(\mathcal{M}))^{\mathfrak{s}} \subset (\mathrm{Sym}^{\cdot}(\overline{\mathcal{M}}))^{\mathfrak{s}} \subset (\mathrm{Sym}^{\cdot}(\overline{\mathcal{M}}))^{\mathfrak{t}}$. In other words, if $\mathcal{S}$ is contained in a positive system then $(\mathrm{Sym}^{\cdot}(\overline{\mathcal{M}}))^{\mathfrak{s}} = \mathbf{C}$.

The goal of this note is to investigate whether either of the conditions $(\mathrm{Sym}^{\cdot}(\mathcal{M}))^{\mathfrak{s}} = \mathbf{C}$ or $(\mathrm{Sym}^{\cdot}(\overline{\mathcal{M}}))^{\mathfrak{s}} = \mathbf{C}$ is sufficient for the existence of a positive system $\mathcal{R}^+$ containing $\mathcal{M}$. It turns out that $(\mathrm{Sym}^{\cdot}(\mathcal{M}))^{\mathfrak{s}} = \mathbf{C}$ is sufficient if and only if $\mathfrak{g}$ is of type A or D and $(\mathrm{Sym}^{\cdot}(\overline{\mathcal{M}}))^{\mathfrak{s}} = \mathbf{C}$ is sufficient if and only if $\mathfrak{g}$ is classical or $\mathfrak{g} = \mathrm{G}_2$.

Using the connection between positive systems and linear functions φ (valid when $\mathfrak{t} = \mathcal{Z}(\mathfrak{s})$), finding a positive system containing $\mathcal{M}$ is the same as finding a parabolic subalgebra $\mathfrak{p}_{\mathcal{M}}$ containing $\mathcal{M}$ with reductive part $\mathfrak{s}$, and we will state our main result in this form. We will also state whether $\mathcal{S}$ is saturated or not, rather than using the notation $\overline{\mathcal{M}}$. In the general case when $\mathfrak{t} \neq \mathcal{Z}(\mathfrak{s})$, the existence of positive systems containing $\mathcal{M}$ is not equivalent to the existence of such a parabolic $\mathfrak{p}_{\mathcal{M}}$. However, our result, as stated below in terms of $\mathfrak{p}_{\mathcal{M}}$, is still valid in this case.

1.5. Main Theorem: Let $\mathfrak{g}$ be a simple Lie algebra, $\mathfrak{t} \subset \mathfrak{g}$ a toral subalgebra, $\mathfrak{s}$ the centralizer of $\mathfrak{t}$, $\mathcal{R}$ the set of $\mathfrak{t}$-roots, $\mathcal{S} \subset \mathcal{R}$, and set $\mathcal{M} = \oplus_{\nu\in\mathcal{S}} \mathfrak{g}^\nu$.

- (*a*) Assume that $(\mathrm{Sym}^{\cdot}(\mathcal{M}))^{\mathfrak{s}} = \mathbf{C}$. If $\mathfrak{g}$ is of type A or D or if $\mathcal{S}$ is saturated and $\mathfrak{g}$ is of type B, C, or G_2 then there exists a parabolic subalgebra $\mathfrak{p}_{\mathcal{M}}$ with reductive part $\mathfrak{s}$ such that $\mathcal{M} \subset \mathfrak{p}_{\mathcal{M}}$.
- (*b*) If $\mathfrak{g}$ is not of type A or D, there exist $\mathcal{S}$ satisfying the condition that $(\mathrm{Sym}^{\cdot}(\mathcal{M}))^{\mathfrak{s}} = \mathbf{C}$ such that no such parabolic $\mathfrak{p}_{\mathcal{M}}$ exists. Moreover, if $\mathfrak{g}$ is F_4, E_6, E_7, or E_8, then $\mathcal{S}$ can be chosen to be saturated.

1.6. Reduction to $\mathfrak{t} = \mathcal{Z}(\mathfrak{s})$. In the main theorem we do not require that $\mathfrak{t} = \mathcal{Z}(\mathfrak{s})$. However, the general case reduces to this case as follows: Set $\mathfrak{t}' := \mathcal{Z}(\mathfrak{s})$ and let $\mathcal{R}'$ be the set of $\mathfrak{t}'$-roots. The natural projection $\pi : (\mathfrak{t}')^* \to \mathfrak{t}^*$ induces a surjection of $\mathcal{R}'$ onto $\mathcal{R}$. Set $\mathcal{S}' := \pi^{-1}(\mathcal{S})$ and notice that

$$\mathcal{M} = \oplus_{\nu\in\mathcal{S}} \mathfrak{g}^\nu = \oplus_{\nu'\in\mathcal{S}'} \mathfrak{g}^{\nu'},$$

and that if $\mathcal{S}$ is saturated, so is $\mathcal{S}'$. Moreover, the centralizer of $\mathfrak{t}'$ is again $\mathfrak{s}$. Thus in proving that $(\mathrm{Sym}^{\cdot}(\mathcal{M}))^{\mathfrak{s}} = \mathbf{C}$ is a sufficient condition we may assume that $\mathfrak{t} = \mathcal{Z}(\mathfrak{s})$. In the cases when we are proving that $(\mathrm{Sym}^{\cdot}(\mathcal{M}))^{\mathfrak{s}} = \mathbf{C}$ is not sufficient, we provide examples in which $\mathfrak{t} = \mathcal{Z}(\mathfrak{s})$.

For the rest of the paper we assume that $\mathfrak{t} = \mathcal{Z}(\mathfrak{s})$.

1.7. Organization and Conventions. In section 2 we describe explicitly all $\mathfrak{t}$-root systems and the respective $\mathfrak{t}$-root spaces for each of the classical simple Lie algebras. In section 3 we first prove the existence of $\mathfrak{p}_{\mathcal{M}}$ when $\mathfrak{g}$ is classical and $\mathcal{S}$ is saturated. We then handle the case of non-saturated $\mathcal{S}$ in types A and D, and finish the section by giving examples in types B and C of non-saturated $\mathcal{S}$ satisfying the condition $(\mathrm{Sym}^{\cdot}(\mathcal{M}))^{\mathfrak{s}} = \mathbf{C}$ for which no parabolic subalgebra $\mathfrak{p}_{\mathcal{M}}$ exists. In section 4 we first treat the case when $\mathfrak{g}$ is of type G_2, proving the result when $\mathcal{S}$ is saturated and giving an example where $\mathcal{S}$ is non-saturated. We then construct examples in types F_4, E_6, E_7, and E_8 of saturated $\mathcal{S}$

for which $(\mathrm{Sym}^{\cdot}(\mathcal{M}))^{\mathfrak{s}} = \mathbf{C}$ and for which no parabolic subalgebra $\mathfrak{p}_{\mathcal{M}}$ exists. That is, in section 3 we establish all parts of the theorem dealing with classical Lie algebras, and in section 4 we establish all parts dealing with the exceptional Lie algebras.

Throughout the paper we work over the field of complex numbers $\mathbf{C}$. All Lie algebras, modules, etc., are over $\mathbf{C}$ unless explicitly stated otherwise. The notation $\subset$ includes the possibility of equality.

2. $\mathfrak{t}$-ROOTS AND $\mathfrak{t}$-ROOT SPACES FOR CLASSICAL LIE ALGEBRAS $\mathfrak{g}$.

2.1. First we describe the parabolic subalgebras and the corresponding sets $\mathcal{R}$ for the classical Lie algebras. For convenience of notation we will work with the reductive Lie algebra $\mathfrak{gl}_n$ instead of $\mathfrak{sl}_n$. For the rest of this section $\mathfrak{g}$ is a classical simple Lie algebra of type B, C, or D or $\mathfrak{g} = \mathfrak{gl}_n$. Moreover, we fix a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. For a comprehensive source on simple complex Lie algebras we refer the reader to [B]. For a treatment of parabolic subalgebras of $\mathfrak{g}$ containing a fixed Cartan subalgebra $\mathfrak{h}$, the reader may also consult [DP].

2.2. Let $\mathcal{P} = \{\mathrm{I}_1, \dots, \mathrm{I}_k\}$ be a partition of $\{1, \dots, n\}$. We say that $\mathcal{P}$ is *totally ordered* if we have given a total order on the set $\{\mathrm{I}_1, \dots, \mathrm{I}_k\}$. We write $\mathcal{P}(i)$ for the part of $\mathcal{P}$ which contains i. The inequalities $\mathcal{P}(i) \prec \mathcal{P}(j)$ and $\mathcal{P}(i) \preceq \mathcal{P}(j)$ are taken in the total order of the parts of $\mathcal{P}$. For the standard basis $\{\varepsilon_1, \dots, \varepsilon_n\}$ of $\mathfrak{h}^*$ we denote the dual basis of $\mathfrak{h}$ by $\{h_1, \dots, h_n\}$. A total order on the set $\{\pm\delta_1, \dots, \pm\delta_k\}$ is *compatible with multiplication by* -1 if, for $x, y \in \{\pm\delta_1, \dots, \pm\delta_k\}$, $x \prec y$ implies $-y \prec -x$. To simplify notation we adopt the convention that B_1, respectively C_1, is a subalgebra of $\mathfrak{g} = \mathrm{B}_n$, respectively $\mathfrak{g} = \mathrm{C}_n$, isomorphic to A_1 and whose roots are short, respectively long roots, of $\mathfrak{g}$. The subalgebras $\mathrm{D}_2 = \mathrm{A}_1 \oplus \mathrm{A}_1$ and $\mathrm{D}_3 = \mathrm{A}_3$ of D_n have similar meaning.

Let $\mathfrak{g}$ be of type $\mathrm{X}_n = \mathrm{A}_n$, B_n, C_n, or D_n and let $\mathfrak{s}$ be a subalgebra of $\mathfrak{g}$ which is the reductive part of a parabolic subalgebra of $\mathfrak{g}$. Every simple ideal of $\mathfrak{s}$ is isomorphic to A_r or X_r for some r. Furthermore, if $\mathfrak{g}$ is not of type A_n, $\mathfrak{s}$ has at most one simple ideal of type X_r. For $\mathfrak{g}$ of type $\mathrm{X}_n = \mathrm{B}_n$, C_n, or D_n the parabolic subalgebras of $\mathfrak{g}$ are split into two types depending on whether their reductive parts contain (Type II) or do not contain (Type I) a simple ideal of type X_r (including B_1, C_1, D_2, or D_3).

In the description of the combinatorics of the simple classical Lie algebras below, the formulas for their parabolic subalgebras $\mathfrak{p}$ containing a fixed reductive part $\mathfrak{s}$ look very uniform (e.g. 11). In some instances this is misleading since the formulas do not explicitly indicate the subalgebra $\mathfrak{s}$ which, however, is an integral part of the structure of $\mathfrak{p}$.

We now list the combinatorial descriptions of the parabolic subalgebras and related data in the classical cases.

2.3. $\mathfrak{g} = \mathfrak{gl}_n$.

1. The roots of $\mathfrak{g}$ are: $\Delta = \{\varepsilon_i - \varepsilon_j \mid 1 \leqslant i \neq j \leqslant n\}$
2. Parabolic subalgebras of $\mathfrak{g}$ are in one-to-one correspondence with:

$$\text{totally ordered partitions } \mathcal{P} = (\mathrm{I}_1, \dots, \mathrm{I}_k) \text{ of } \{1, \dots, n\}.$$

Given a totally ordered partition $\mathcal{P}$,

3. The roots of $\mathfrak{p}_{\mathcal{P}}$ are $\{\varepsilon_i - \varepsilon_j \,|\, i \neq j, \mathcal{P}(i) \preceq \mathcal{P}(j)\}$
4. The roots of $\mathfrak{s}_{\mathcal{P}}$ are $\{\varepsilon_i - \varepsilon_j \,|\, i \neq j, \mathcal{P}(i) = \mathcal{P}(j)\}$
5. $\mathfrak{s}_{\mathcal{P}} = \oplus_i \mathfrak{s}^i_{\mathcal{P}}$, where $\mathfrak{s}^i_{\mathcal{P}} \cong \mathfrak{gl}_{|\mathrm{I}_i|}$;
6. The Cartan subalgebra of $\mathfrak{s}^i_{\mathcal{P}}$ is spanned by $\{h_j\}_{j \in \mathrm{I}_i}$
7. The roots of $\mathfrak{s}^i_{\mathcal{P}}$ are $\{\varepsilon_j - \varepsilon_l \,|\, j \neq l \in \mathrm{I}_i\}$.
8. $\mathfrak{t}_{\mathcal{P}}$ has a basis $\{t_1, \ldots, t_k\}$ with $t_i = \frac{1}{|\mathrm{I}_i|} \sum_{j \in \mathrm{I}_i} h_j$

If $\{\delta_1, \ldots, \delta_k\}$ is the basis of $\mathfrak{t}^*$ dual to $\{t_1, \ldots, t_k\}$ then

9. $\mathcal{R} = \{\delta_i - \delta_j \,|\, 1 \leqslant i \neq j \leqslant k\}$.
10. For $\nu = \delta_i - \delta_j \in \mathcal{R}$, $\mathfrak{g}^\nu \cong \mathrm{V}_i \otimes \mathrm{V}_j^*$, where V_i and V_j^* are the natural $\mathfrak{s}^i_{\mathcal{P}}$-module and the dual of the natural $\mathfrak{s}^j_{\mathcal{P}}$-module respectively, all other factors of $\mathfrak{s}_{\mathcal{P}}$ acting trivially.
11. The parabolic subalgebras of $\mathfrak{g}$ whose reductive part is $\mathfrak{s}_{\mathcal{P}}$ are in a bijection with the ordered partitions $\mathcal{Q}$ of $\{1, \ldots, n\}$ whose parts are the same as the parts of $\mathcal{P}$ or, equivalently, with total orders on the set $\{\delta_1, \ldots, \delta_k\}$.

2.4. $\mathfrak{g} = \mathrm{B}_n$

1. The roots of $\mathfrak{g}$ are: $\Delta = \{\pm\varepsilon_i \pm \varepsilon_j, \pm\varepsilon_i \,|\, 1 \leqslant i \neq j \leqslant n\}$.
2. Parabolic subalgebras of $\mathfrak{g}$ are in one-to-one correspondence with:

Type I: pairs $(\mathcal{P}, \sigma)$, where $\mathcal{P} = (\mathrm{I}_1, \ldots, \mathrm{I}_k)$ is a totally ordered partition of $\{1, \ldots, n\}$ and $\sigma \colon \{1, \ldots, n\} \to \{\pm 1\}$ is a choice of signs.

Type II: pairs $(\mathcal{P}, \sigma)$, where $\mathcal{P} = (\mathrm{I}_0, \mathrm{I}_1, \ldots, \mathrm{I}_k)$ is a totally ordered partition of $\{1, \ldots, n\}$ with largest element I_0 and $\sigma \colon \{1, \ldots, n\} \backslash \mathrm{I}_0 \to \{\pm 1\}$ is a choice of signs.

<u>In Type I:</u>

3. The roots of $\mathfrak{p}_{(\mathcal{P},\sigma)}$ are
$$\{\sigma(i)\varepsilon_i - \sigma(j)\varepsilon_j \;\mid\; i \neq j, \mathcal{P}(i) \preceq \mathcal{P}(j)\} \cup \{\sigma(i)\varepsilon_i + \sigma(j)\varepsilon_j, \sigma(i)\varepsilon_i \;\mid\; i \neq j\}$$
4. The roots of $\mathfrak{s}_{(\mathcal{P},\sigma)}$ are $\{\sigma(i)\varepsilon_i - \sigma(j)\varepsilon_j \,|\, i \neq j, \mathcal{P}(i) = \mathcal{P}(j)\}$.
5. $\mathfrak{s}_{(\mathcal{P},\sigma)} = \oplus_i \mathfrak{s}^i_{(\mathcal{P},\sigma)}$, where $\mathfrak{s}^i_{(\mathcal{P},\sigma)} \cong \mathfrak{gl}_{|\mathrm{I}_i|}$.
6. The Cartan subalgebra of $\mathfrak{s}^i_{\mathcal{P}}$ is spanned by $\{\sigma(j)h_j\}_{j \in \mathrm{I}_i}$
7. The roots of $\mathfrak{s}^i_{\mathcal{P}}$ are $\{\sigma(j)\varepsilon_j - \sigma(l)\varepsilon_l \,|\, j \neq l \in \mathrm{I}_i\}$.
8. $\mathfrak{t}_{(\mathcal{P},\sigma)}$ has a basis $\{t_1, \ldots, t_k\}$ with $t_i = \frac{1}{|\mathrm{I}_i|} \sum_{j \in \mathrm{I}_i} \sigma(j) h_j$.

If $\{\delta_1, \ldots, \delta_k\}$ the basis of $\mathfrak{t}^*$ dual to $\{t_1, \ldots, t_k\}$ then

9. $\mathcal{R} = \{\pm\delta_i \pm \delta_j, \pm\delta_i \,|\, 1 \leqslant i \neq j \leqslant k\} \cup \{\pm 2\delta_i \,|\, |\mathrm{I}_i| > 1\}$.
10. For $\nu \in \mathcal{R}$,
    (a) $\mathfrak{g}^\nu \cong \mathrm{V}_i^\pm \otimes \mathrm{V}_j^\pm$ if $\nu = \pm\delta_i \pm \delta_j$,
    (b) $\mathfrak{g}^\nu \cong \mathrm{V}_i^\pm$ if $\nu = \pm\delta_i$, and
    (c) $\mathfrak{g}^\nu \cong \Lambda^2 \mathrm{V}_i^\pm$ if $\nu = \pm 2\delta_i$,

where V_i^+ and V_i^- respectively are the natural $\mathfrak{s}^i_{(\mathcal{P},\sigma)}$-module and its dual, and all other factors of $\mathfrak{s}_{(\mathcal{P},\sigma)}$ act trivially.

11. The parabolic subalgebras of $\mathfrak{g}$ whose reductive part is $\mathfrak{s}_{\mathcal{P},\sigma}$ are in a bijection with the pairs $(\mathcal{Q},\tau)$ such that the parts of $\mathcal{Q}$ are the same as the parts of $\mathcal{P}$ and $\sigma_{|I_i} = \pm\tau_{|I_i}$ for every part I_i or, equivalently, with total orders on the set $\{\pm\delta_1,\ldots,\pm\delta_k\}$ compatible with multiplication by -1.

<u>In Type II:</u>

3. The roots of $\mathfrak{p}_{(\mathcal{P},\sigma)}$ are

$$\{\sigma(i)\varepsilon_i-\sigma(j)\varepsilon_j\,|\,i\neq j,\mathcal{P}(i)\preceq\mathcal{P}(j)\prec I_0\}\cup\{\pm\varepsilon_i\pm\varepsilon_j,\pm\varepsilon_i\,|\,i\neq j,i\in I_0,j\in I_0\}$$
$$\cup\{\sigma(i)\varepsilon_i+\sigma(j)\varepsilon_j,\sigma(i)\varepsilon_i\,|\,i\neq j,i\notin I_0,j\notin I_0\}\cup\{\sigma(i)\varepsilon_i\pm\varepsilon_j\,|\,i\notin I_0,j\in I_0\}$$

4. The roots of $\mathfrak{s}_{(\mathcal{P},\sigma)}$ are

$$\{\sigma(i)\varepsilon_i-\sigma(j)\varepsilon_j\,|\,i\neq j,\mathcal{P}(i)=\mathcal{P}(j)\prec I_0\}\cup\{\pm\varepsilon_i\pm\varepsilon_j,\pm\varepsilon_i\,|\,i\neq j\in I_0\}.$$

5. $\mathfrak{s}_{(\mathcal{P},\sigma)}=\oplus_i\,\mathfrak{s}^i_{(\mathcal{P},\sigma)}$, where $\mathfrak{s}^0_{(\mathcal{P},\sigma)}\cong B_{|I_0|}$ and $\mathfrak{s}^i_{(\mathcal{P},\sigma)}\cong\mathfrak{gl}_{|I_i|}$ for $i>0$.
6. The Cartan subalgebra of $\mathfrak{s}^i_{\mathcal{P}}$ is spanned by $\{h_j\}_{j\in I_0}$ for $i=0$ and $\{\sigma(j)h_j\}_{j\in I_i}$ for $i>0$.
7. The roots of $\mathfrak{s}^i_{\mathcal{P}}$ are $\{\pm\varepsilon_j\pm\varepsilon_l,\pm\varepsilon_j\,|\,j\neq l\in I_0\}$ for $i=0$ and $\{\sigma(j)\varepsilon_j-\sigma(l)\varepsilon_l\,|\,j\neq l\in I_i\}$ for $i>0$.
8. $\mathfrak{t}_{(\mathcal{P},\sigma)}$ has a basis $\{t_1,\ldots,t_k\}$ with $t_i=\frac{1}{|I_i|}\sum_{j\in I_i}\sigma(j)h_j$.

If $\{\delta_1,\ldots,\delta_k\}$ is the basis of $\mathfrak{t}^*$ dual to $\{t_1,\ldots,t_k\}$ then

9. $\mathcal{R}=\{\pm\delta_i\pm\delta_j,\pm\delta_i\,|\,1\leqslant i\neq j\leqslant k\}\cup\{\pm2\delta_i\,|\,|I_i|>1\}$.
10. For $\nu\in\mathcal{R}$,
    (a) $\mathfrak{g}^\nu\cong V_i^\pm\otimes V_j^\pm$ if $\nu=\pm\delta_i\pm\delta_j$,
    (b) $\mathfrak{g}^\nu\cong V_i^\pm\otimes V_0$ if $\nu=\pm\delta_i$, and
    (c) $\mathfrak{g}^\nu\cong\Lambda^2V_i^\pm$ if $\nu=\pm2\delta_i$

    where V_i^+ and V_i^- denote the natural $\mathfrak{s}^i_{(\mathcal{P},\sigma)}$-module and its dual respectively for $i>0$, V_0 denotes the natural $\mathfrak{s}^0_{(\mathcal{P},\sigma)}$-module, and all other factors of $\mathfrak{s}_{(\mathcal{P},\sigma)}$ act trivially. Note that, if $\mathfrak{s}_{(\mathcal{P},\sigma)}=B_1\cong\mathfrak{sl}_2$, then V_0 is the three dimensional irreducible $\mathfrak{s}_{(\mathcal{P},\sigma)}$-module.

11. The parabolic subalgebras of $\mathfrak{g}$ whose reductive part is $\mathfrak{s}_{\mathcal{P},\sigma}$ are in a bijection with the pairs $(\mathcal{Q},\tau)$ such that the parts of $\mathcal{Q}$ are the same as the parts of $\mathcal{P}$, I_0 is the largest element of $\mathcal{Q}$, and $\sigma_{|I_i}=\pm\tau_{|I_i}$ for every part $I_i\neq I_0$ or, equivalently, with total orders on the set $\{\pm\delta_1,\ldots,\pm\delta_k\}$ compatible with multiplication by -1.

2.5. $\mathfrak{g}=C_n$

1. The roots of $\mathfrak{g}$ are: $\Delta=\{\pm\varepsilon_i\pm\varepsilon_j,\pm2\varepsilon_i\,|\,1\leqslant i\neq j\leqslant n\}$.
2. Parabolic subalgebras of $\mathfrak{g}$ are in one-to-one correspondence with:

Type I: pairs $(\mathcal{P},\sigma)$, where $\mathcal{P}=(I_1,\ldots,I_k)$ is a totally ordered partition of $\{1,\ldots,n\}$ and $\sigma\colon\{1,\ldots,n\}\to\{\pm1\}$ is a choice of signs.

Type II: pairs $(\mathcal{P},\sigma)$, where $\mathcal{P}=(\mathrm{I}_0,\mathrm{I}_1,\ldots,\mathrm{I}_k)$ is a totally ordered partition of $\{1,\ldots,n\}$ with largest element I_0 and $\sigma\colon\{1,\ldots,n\}\backslash\mathrm{I}_0\to\{\pm 1\}$ is a choice of signs.

In Type I:

3. The roots of $\mathfrak{p}_{(\mathcal{P},\sigma)}$ are
$$\{\sigma(i)\varepsilon_i-\sigma(j)\varepsilon_j\,|\,i\neq j,\mathcal{P}(i)\preceq\mathcal{P}(j)\}\cup\{\sigma(i)\varepsilon_i+\sigma(j)\varepsilon_j,2\sigma(i)\varepsilon_i\,|\,i\neq j\}$$
4. The roots of $\mathfrak{s}_{(\mathcal{P},\sigma)}$ are $\{\sigma(i)\varepsilon_i-\sigma(j)\varepsilon_j\,|\,i\neq j,\mathcal{P}(i)=\mathcal{P}(j)\}$.
5. $\mathfrak{s}_{(\mathcal{P},\sigma)}=\oplus_i\,\mathfrak{s}^i_{(\mathcal{P},\sigma)}$, where $\mathfrak{s}^i_{(\mathcal{P},\sigma)}\cong\mathfrak{gl}_{|\mathrm{I}_i|}$.
6. The Cartan subalgebra of $\mathfrak{s}^i_{\mathcal{P}}$ is spanned by $\{\sigma(j)h_j\}_{j\in\mathrm{I}_i}$.
7. The roots of $\mathfrak{s}^i_{(\mathcal{P},\sigma)}$ are $\{\sigma(j)\varepsilon_j-\sigma(l)\varepsilon_l\,|\,j\neq l\in\mathrm{I}_i\}$.
8. $\mathfrak{t}_{(\mathcal{P},\sigma)}$ has a basis $\{t_1,\ldots,t_k\}$ with $t_i=\frac{1}{|\mathrm{I}_i|}\sum_{j\in\mathrm{I}_i}\sigma(j)h_j$.

If $\{\delta_1,\ldots,\delta_k\}$ is the basis of $\mathfrak{t}^*$ dual to $\{t_1,\ldots,t_k\}$ then

9. $\mathcal{R}=\{\pm\delta_i\pm\delta_j,\pm 2\delta_i\,|\,1\leqslant i\neq j\leqslant k\}$.
10. For $\nu\in\mathcal{R}$,
    (a) $\mathfrak{g}^\nu\cong\mathrm{V}_i^\pm\otimes\mathrm{V}_j^\pm$ if $\nu=\pm\delta_i\pm\delta_j$.
    (b) $\mathfrak{g}^\nu\cong\mathrm{Sym}^2\,\mathrm{V}_i^\pm$ if for $\nu=\pm 2\delta_i$.

    where V_i^+ and V_i^- are the natural $\mathfrak{s}^i_{(\mathcal{P},\sigma)}$-module and its dual, and all other factors of $\mathfrak{s}_{(\mathcal{P},\sigma)}$ act trivially.
11. The parabolic subalgebras of $\mathfrak{g}$ whose reductive part is $\mathfrak{s}_{\mathcal{P},\sigma}$ are in a bijection with the pairs $(\mathcal{Q},\tau)$ such that the parts of $\mathcal{Q}$ are the same as the parts of $\mathcal{P}$ and $\sigma_{|\mathrm{I}_i}=\pm\tau_{|\mathrm{I}_i}$ for every part I_i or, equivalently, with total orders on the set $\{\pm\delta_1,\ldots,\pm\delta_k\}$ compatible with multiplication by -1.

In Type II:

3. The roots of $\mathfrak{p}_{(\mathcal{P},\sigma)}$ are
$$\{\sigma(i)\varepsilon_i-\sigma(j)\varepsilon_j\,|\,i\neq j,\mathcal{P}(i)\preceq\mathcal{P}(j)\prec\mathrm{I}_0\}\cup\{\pm\varepsilon_i\pm\varepsilon_j,\pm 2\varepsilon_i\,|\,i\neq j,i\in\mathrm{I}_0,j\in\mathrm{I}_0\}\cup$$
$$\{\sigma(i)\varepsilon_i+\sigma(j)\varepsilon_j,\sigma(i)2\varepsilon_i\,|\,i\neq j,i\notin\mathrm{I}_0,j\notin\mathrm{I}_0\}\cup\{\sigma(i)\varepsilon_i\pm\varepsilon_j\,|\,i\notin\mathrm{I}_0,j\in\mathrm{I}_0\}$$
4. The roots of $\mathfrak{s}_{(\mathcal{P},\sigma)}$ are
$$\{\sigma(i)\varepsilon_i-\sigma(j)\varepsilon_j\,|\,i\neq j,\mathcal{P}(i)=\mathcal{P}(j)\prec\mathrm{I}_0\}\cup\{\pm\varepsilon_i\pm\varepsilon_j,\pm 2\varepsilon_i\,|\,i\neq j\in\mathrm{I}_0\}.$$
5. $\mathfrak{s}_{(\mathcal{P},\sigma)}=\oplus_{i=0}^k\,\mathfrak{s}^i_{(\mathcal{P},\sigma)}$, where $\mathfrak{s}^0_{(\mathcal{P},\sigma)}\cong\mathrm{C}_{|\mathrm{I}_0|}$ and $\mathfrak{s}^i_{(\mathcal{P},\sigma)}\cong\mathfrak{gl}_{|\mathrm{I}_i|}$ for $i>0$.
6. The Cartan subalgebra of $\mathfrak{s}^i_{(\mathcal{P},\sigma)}$ is spanned by $\{h_j\}_{j\in\mathrm{I}_0}$ for $i=0$ and $\{\sigma(j)h_j\}_{j\in\mathrm{I}_i}$ for $i>0$.
7. The roots of $\mathfrak{s}^i_{(\mathcal{P},\sigma)}$ are $\{\pm\varepsilon_j\pm\varepsilon_l,\pm 2\varepsilon_j\,|\,j\neq l\in\mathrm{I}_0\}$ for $i=0$ and $\{\sigma(j)\varepsilon_j-\sigma(l)\varepsilon_l\,|\,j\neq l\in\mathrm{I}_i\}$ for $i>0$.
8. $\mathfrak{t}_{(\mathcal{P},\sigma)}$ has a basis $\{t_1,\ldots,t_k\}$ with $t_i=\frac{1}{|\mathrm{I}_i|}\sum_{j\in\mathrm{I}_i}\sigma(j)h_j$

If $\{\delta_1,\ldots,\delta_k\}$ is the basis of $\mathfrak{t}^*$ dual to $\{t_1,\ldots,t_k\}$ then

9. $\mathcal{R}=\{\pm\delta_i\pm\delta_j,\pm\delta_i,\pm 2\delta_i\,|\,1\leqslant i\neq j\leqslant k\}$.
10. For $\nu\in\mathcal{R}$,
    (a) $\mathfrak{g}^\nu\cong\mathrm{V}_i^\pm\otimes\mathrm{V}_j^\pm$ if $\nu=\pm\delta_i\pm\delta_j$,
    (b) $\mathfrak{g}^\nu\cong\mathrm{V}_i^\pm\otimes\mathrm{V}_0$ if $\nu=\pm\delta_i$,

(c) $\mathfrak{g}^\nu$ is isomorphic to $\mathrm{Sym}^2\,\mathrm{V}_i^\pm$ if $\nu = \pm 2\delta_i$,

where V_i^+ and V_i^- denote the natural $\mathfrak{s}^i_{(\mathcal{P},\sigma)}$-module and its dual for $i > 0$, V_0 is the natural $\mathfrak{s}^0_{(\mathcal{P},\sigma)}$-module, and where all other factors of $\mathfrak{s}_{(\mathcal{P},\sigma)}$ act trivially. Note that, if $\mathfrak{s}_{(\mathcal{P},\sigma)} = \mathrm{C}_1 \cong \mathfrak{sl}_2$, then V_0 is the two dimensional irreducible $\mathfrak{s}_{(\mathcal{P},\sigma)}$-module.

11. The parabolic subalgebras of $\mathfrak{g}$ whose reductive part is $\mathfrak{s}_{\mathcal{P},\sigma}$ are in a bijection with the pairs $(\mathcal{Q},\tau)$ such that the parts of $\mathcal{Q}$ are the same as the parts of $\mathcal{P}$, I_0 is the largest element of $\mathcal{Q}$, and $\sigma_{|\mathrm{I}_i} = \pm\tau_{|\mathrm{I}_i}$ for every part $\mathrm{I}_i \neq \mathrm{I}_0$ or, equivalently, with total orders on the set $\{\pm\delta_1, \ldots, \pm\delta_k\}$ compatible with multiplication by -1.

2.6. $\mathfrak{g} = \mathrm{D}_n$.

1. The roots of $\mathfrak{g}$ are: $\Delta = \{\pm\varepsilon_i \pm \varepsilon_j, \,|\, 1 \leqslant i \neq j \leqslant n\}$.
2. Parabolic subalgebras of $\mathfrak{g}$ are determined by:

Type I: pairs $(\mathcal{P},\sigma)$, where $\mathcal{P} = (\mathrm{I}_0, \mathrm{I}_1, \ldots, \mathrm{I}_k)$ is a totally ordered partition of $\{1,\ldots,n\}$ and $\sigma : \{1,\ldots,n\} \to \{\pm 1\}$ is a choice of signs.

Two pairs $(\mathcal{P}',\sigma')$ and $(\mathcal{P}'',\sigma'')$ determine the same parabolic subalgebra if and only if $\mathcal{P}'$ and $\mathcal{P}''$ are the same ordered partitions whose maximal part I_0 contains one element and σ' and σ'' coincide on $\{1,\ldots,n\}\backslash\mathrm{I}_0$.

Type II: pairs $(\mathcal{P},\sigma)$, where $\mathcal{P} = (\mathrm{I}_0, \mathrm{I}_1, \ldots, \mathrm{I}_k)$ is a totally ordered partition of $\{1,\ldots,n\}$ with largest element I_0 such that $|\mathrm{I}_0| \geqslant 2$ and $\sigma : \{1,\ldots,n\}\backslash\mathrm{I}_0 \to \{\pm 1\}$ is a choice of signs.

<u>In Type I:</u>

3. The roots of $\mathfrak{p}_{(\mathcal{P},\sigma)}$ are $\{\sigma(i)\varepsilon_i - \sigma(j)\varepsilon_j \,|\, i \neq j, \mathcal{P}(i) \preceq \mathcal{P}(j)\} \cup \{\sigma(i)\varepsilon_i + \sigma(j)\varepsilon_j \,|\, i \neq j\}$
4. The roots of $\mathfrak{s}_{(\mathcal{P},\sigma)}$ are $\{\sigma(i)\varepsilon_i - \sigma(j)\varepsilon_j \,|\, i \neq j, \mathcal{P}(i) = \mathcal{P}(j)\}$.
5. $\mathfrak{s}_{(\mathcal{P},\sigma)} = \oplus_i\, \mathfrak{s}^i_{(\mathcal{P},\sigma)}$, where $\mathfrak{s}^i_{(\mathcal{P},\sigma)} \cong \mathfrak{gl}_{|\mathrm{I}_i|}$.
6. The Cartan subalgebra of $\mathfrak{s}^i_{\mathcal{P}}$ is spanned by $\{\sigma(j)h_j\}_{j\in\mathrm{I}_i}$
7. The roots of $\mathfrak{s}^i_{(\mathcal{P},\sigma)}$ are $\{\sigma(j)\varepsilon_j - \sigma(l)\varepsilon_l \,|\, j \neq l \in \mathrm{I}_i\}$.
8. $\mathfrak{t}_{(\mathcal{P},\sigma)}$ has a basis $\{t_1,\ldots,t_k\}$ with $t_i = \frac{1}{|\mathrm{I}_i|}\sum_{j\in\mathrm{I}_i}\sigma(j)h_j$.

If $\{\delta_1,\ldots,\delta_k\}$ is the basis of $\mathfrak{t}^*$ dual to $\{t_1,\ldots,t_k\}$ then

9. $\mathcal{R} = \{\pm\delta_i \pm \delta_j |\, 1 \leqslant i \neq j \leqslant k\} \cup \{\pm 2\delta_i \,|\, |\mathrm{I}_i| > 1\}$.
10. For $\nu \in \mathcal{R}$,
   (a) $\mathfrak{g}^\nu \cong \mathrm{V}_i^\pm \otimes \mathrm{V}_j^\pm$ if $\nu = \pm\delta_i \pm \delta_j$.
   (b) $\mathfrak{g}^\nu \cong \Lambda^2 \mathrm{V}_i^\pm$ if $\nu = \pm 2\delta_i$.

   where V_i^+ and V_i^- are the natural $\mathfrak{s}^i_{(\mathcal{P},\sigma)}$-module and its dual, and all other factors of $\mathfrak{s}_{(\mathcal{P},\sigma)}$ act trivially.
11. Every parabolic subalgebra of $\mathfrak{g}$ whose reductive part is $\mathfrak{s}_{\mathcal{P},\sigma}$ corresponds to a pair $(\mathcal{Q},\tau)$ such that the parts of $\mathcal{Q}$ are the same as the parts of $\mathcal{P}$ and $\sigma_{|\mathrm{I}_i} = \pm\tau_{|\mathrm{I}_i}$ for every part I_i or, equivalently, to a total order on the set $\{\pm\delta_1,\ldots,\pm\delta_k\}$ compatible with multiplication by -1. Note that this correspondence is not bijective since two different total orders may determine the same parabolic subalgebra.

In Type II:

3. Roots of $\mathfrak{p}_{(\mathcal{P},\sigma)} =$

$$\{\sigma(i)\varepsilon_i - \sigma(j)\varepsilon_j \;\mid\; i \neq j, \mathcal{P}(i) \preceq \mathcal{P}(j) \prec \mathrm{I}_0\} \cup \{\pm\varepsilon_i \pm \varepsilon_j \,|\, i \neq j, i \in \mathrm{I}_0, j \in \mathrm{I}_0\}$$
$$\cup\{\sigma(i)\varepsilon_i + \sigma(j)\varepsilon_j \,|\, i \neq j, i \notin \mathrm{I}_0, j \notin \mathrm{I}_0\} \cup \{\sigma(i)\varepsilon_i \pm \varepsilon_j \,|\, i \notin \mathrm{I}_0, j \in \mathrm{I}_0\}$$

4. Roots of $\mathfrak{s}_{(\mathcal{P},\sigma)} = \{\sigma(i)\varepsilon_i - \sigma(j)\varepsilon_j \,|\, i \neq j, \mathcal{P}(i) = \mathcal{P}(j) \prec \mathrm{I}_0\} \cup \{\pm\varepsilon_i \pm \varepsilon_j \,|\, i \neq j \in \mathrm{I}_0\}$.
5. $\mathfrak{s}_{(\mathcal{P},\sigma)} = \oplus_{i=0}^{k} \mathfrak{s}^i_{(\mathcal{P},\sigma)}$, where $\mathfrak{s}^0_{(\mathcal{P},\sigma)} \cong \mathrm{D}_{|\mathrm{I}_0|}$ and $\mathfrak{s}^i_{(\mathcal{P},\sigma)} \cong \mathfrak{gl}_{|\mathrm{I}_i|}$ for $i > 0$.
6. Cartan subalgebra of $\mathfrak{s}^i_{(\mathcal{P},\sigma)}$ is spanned by $\{h_j\}_{j\in\mathrm{I}_0}$ for $i = 0$ and by $\{\sigma(j)h_j\}_{j\in\mathrm{I}_i}$ for $i > 0$;
7. roots of $\mathfrak{s}^i_{(\mathcal{P},\sigma)}$ are $\{\pm\varepsilon_j \pm \varepsilon_l \,|\, j \neq l \in \mathrm{I}_0\}$ for $i = 0$ and $\{\sigma(j)\varepsilon_j - \sigma(l)\varepsilon_l \,|\, j \neq l \in \mathrm{I}_i\}$ for $i > 0$.
8. $\mathfrak{t}_{(\mathcal{P},\sigma)}$ has a basis $\{t_1, \ldots, t_k\}$ with $t_i = \frac{1}{|\mathrm{I}_i|}\sum_{j\in\mathrm{I}_i} \sigma(j)h_j$.

If $\{\delta_1, \ldots, \delta_k\}$ is the basis of $\mathfrak{t}^*$ dual to $\{t_1, \ldots, t_k\}$ then

9. $\mathcal{R} = \{\pm\delta_i \pm \delta_j, \pm\delta_i \,|\, 1 \leqslant i \neq j \leqslant k\} \cup \{\pm 2\delta_i \,|\, |\mathrm{I}_i| > 1\}$.
10. For $\nu \in \mathcal{R}$,
    (a) $\mathfrak{g}^\nu \cong \mathrm{V}_i^\pm \otimes \mathrm{V}_j^\pm$ if $\nu = \pm\delta_i \pm \delta_j$,
    (b) $\mathfrak{g}^\nu \cong \mathrm{V}_i^\pm \otimes \mathrm{V}_0$ if $\nu = \pm\delta_i$,
    (c) $\mathfrak{g}^\nu \cong \Lambda^2\mathrm{V}_i^\pm$ if $\nu = \pm 2\delta_i$,

    where V_i^+ and V_i^- denote the natural $\mathfrak{s}^i_{(\mathcal{P},\sigma)}$-module and its dual for $i > 0$, V_0 is the natural $\mathfrak{s}^0_{(\mathcal{P},\sigma)}$-module, and where all other factors of $\mathfrak{s}_{(\mathcal{P},\sigma)}$ act trivially. Note that, if $\mathfrak{s}_{(\mathcal{P},\sigma)} = \mathrm{D}_2 \cong \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$, then V_0 is the (external) tensor product of two two-dimensional irreducible $\mathfrak{sl}_2$-modules; if $\mathfrak{s}_{(\mathcal{P},\sigma)} = \mathrm{D}_3 \cong \mathfrak{sl}_4$, then V_0 the six dimensional irreducible $\mathfrak{s}_{(\mathcal{P},\sigma)}$-module which is the second exterior power of the natural representation of $\mathfrak{sl}_4$.
11. The parabolic subalgebras of $\mathfrak{g}$ whose reductive part is $\mathfrak{s}_{(\mathcal{P},\sigma)}$ are in a bijection with the pairs $(\mathcal{Q}, \tau)$ such that the parts of $\mathcal{Q}$ are the same as the parts of $\mathcal{P}$, I_0 is the largest element of $\mathcal{Q}$, and $\sigma_{|\mathrm{I}_i} = \pm\tau_{|\mathrm{I}_i}$ for every part I_i or, equivalently, with total orders on the set $\{\pm\delta_1, \ldots, \pm\delta_k\}$.

3. Proof of the Main Theorem when $\mathfrak{g}$ is classical.

3.1. Existence of $\mathfrak{p}_{\mathcal{M}}$ when $\mathcal{S}$ is saturated. The idea is simple: using $\mathcal{S}$ we define a binary relation $\prec$ on the set $\{\delta_1, \ldots, \delta_k\}$ (respectively on $\{\pm\delta_1, \ldots, \pm\delta_k\}$) and using the fact that $(\mathrm{Sym}^{\cdot}(\mathcal{M}))^{\mathfrak{s}} = \mathbf{C}$ we prove that $\prec$ can be extended to a total order (respectively, to a total order compatible with multiplication by -1). The proof follows the same logic in all cases but is least technical in the case when $\mathfrak{g} = \mathfrak{gl}_n$. For clarity of exposition we present the proof for $\mathfrak{g} = \mathfrak{gl}_n$ first. Throughout the proof the partition $\mathcal{P}$ (and the choice of signs σ) are fixed and instead of $\mathfrak{s}_{\mathcal{P}}$ (or $\mathfrak{s}_{(\mathcal{P},\sigma)}$) and $\mathfrak{s}^i_{\mathcal{P}}$ (or $\mathfrak{s}^i_{(\mathcal{P},\sigma)}$) we write $\mathfrak{s}$ and $\mathfrak{s}^i$ respectively.

First we consider the case when $\mathfrak{g} = \mathfrak{gl}_n$. Define a binary relation $\prec$ on $\{\delta_1, \delta_2, \ldots, \delta_k\}$ by setting

$$\textbf{(3.1)} \qquad \delta_i \prec \delta_j \quad \text{if} \quad \nu = \delta_i - \delta_j \in \mathcal{S}.$$

The existence of a parabolic subalgebra $\mathfrak{p}_{\mathcal{M}}$ with reductive part $\mathfrak{s}$ and containing $\mathcal{M}$ is equivalent to the existence of a total order on $\{\delta_1, \delta_2, \dots, \delta_k\}$ which extends $\prec$.

Note that $\prec$ can be extended to a total order on $\{\delta_1, \delta_2, \dots, \delta_k\}$ if and only if there is no cycle

$$\delta_{i_1} \prec \delta_{i_2} \prec \cdots \prec \delta_{i_l} \prec \delta_{i_1}. \tag{3.2}$$

Assume that $\prec$ cannot be extended to a total order on $\{\delta_1, \cdots, \delta_k\}$ and consider a cycle (3.2) of minimal length. Then $\nu_1 = \delta_{i_1} - \delta_{i_2}, \nu_2 = \delta_{i_2} - \delta_{i_3}, \cdots, \nu_l = \delta_{i_l} - \delta_{i_1}$ is a sequence of distinct elements of $\mathcal{S}$. Hence $\mathfrak{g}^{\nu_1} \oplus \mathfrak{g}^{\nu_2} \oplus \cdots \oplus \mathfrak{g}^{\nu_l}$ is a submodule of $\mathcal{M}$ and $\mathrm{Sym}^l(\mathfrak{g}^{\nu_1} \oplus \mathfrak{g}^{\nu_2} \oplus \cdots \oplus \mathfrak{g}^{\nu_l})$ is a submodule of $\mathrm{Sym}^{\cdot}(\mathcal{M})$ containing $\mathfrak{g}^{\nu_1} \otimes \mathfrak{g}^{\nu_2} \otimes \cdots \otimes \mathfrak{g}^{\nu_l}$. On the other hand,

$$\begin{aligned} \mathfrak{g}^{\nu_1} \otimes \mathfrak{g}^{\nu_2} \otimes \cdots \otimes \mathfrak{g}^{\nu_l} &\cong (\mathrm{V}_{i_1} \otimes \mathrm{V}_{i_2}^*) \otimes (\mathrm{V}_{i_2} \otimes \mathrm{V}_{i_3}^*) \otimes \cdots \otimes (\mathrm{V}_{i_l} \otimes \mathrm{V}_{i_1}^*) \\ &\cong (\mathrm{V}_{i_1} \otimes \mathrm{V}_{i_1}^*) \otimes (\mathrm{V}_{i_2} \otimes \mathrm{V}_{i_2}^*) \otimes \cdots \otimes (\mathrm{V}_{i_l} \otimes \mathrm{V}_{i_l}^*), \end{aligned} \tag{3.3}$$

where the lower index of a module shows which component of $\mathfrak{s}$ acts non-trivially on it. Since, for every $1 \leqslant j \leqslant l$, $\mathrm{V}_{i_j} \otimes \mathrm{V}_{i_j}^*$ contains the trivial $\mathfrak{s}^{i_j}$-module, (3.3) shows that $\mathfrak{g}^{\nu_1} \otimes \mathfrak{g}^{\nu_2} \otimes \cdots \otimes \mathfrak{g}^{\nu_l}$ contains the trivial $\mathfrak{s}$-module which contradicts the assumption that $(\mathrm{Sym}^{\cdot}(\mathcal{M}))^{\mathfrak{s}} = \mathbf{C}$. This contradiction shows that $\prec$ can be extended to a total order on the set $\{\delta_1, \cdots, \delta_k\}$, which completes the proof when $\mathfrak{g} = \mathfrak{gl}_n$.

Next we consider the case when $\mathfrak{g} \neq \mathfrak{gl}_n$, i.e., we assume that $\mathfrak{g}$ is a simple classical Lie algebra not of type A. Define a binary relation $\prec$ on $\{\pm, \delta_1, \pm\delta_2, \cdots, \pm\delta_k\}$ by setting

$$\begin{array}{lll} s_i\delta_i \prec s_j\delta_j,\ i \neq j & \text{if} & \nu = s_i\delta_i - s_j\delta_j \in \mathcal{S} \\ s_i\delta_i \prec -s_i\delta_i & \text{if} & \nu = \left\{ \begin{array}{lll} s_i\delta_i & \text{when} & \mathfrak{g} = \mathrm{B}_n \text{ or } \mathfrak{g} = \mathrm{D}_n, \text{ Type II} \\ 2s_i\delta_i & \text{when} & \mathfrak{g} = \mathrm{C}_n \text{ or } \mathfrak{g} = \mathrm{D}_n, \text{ Type I} \end{array} \right. \in \mathcal{S}, \end{array} \tag{3.4}$$

where $s_i, s_j = \pm$. Note that $\prec$ is compatible with multiplication by -1.

The existence of a parabolic subalgebra $\mathfrak{p}_{\mathcal{M}}$ with reductive part $\mathfrak{s}$ and containing $\mathcal{M}$ is equivalent the existence of a total order on $\{\pm\delta_1, \pm\delta_2, \cdots, \pm\delta_k\}$ compatible with multiplication by -1 which extends $\prec$.

Note that $\prec$ can be extended to a total order on $\{\pm\delta_1, \pm\delta_2, \cdots, \pm\delta_k\}$ compatible with multiplication by -1 if and only if there is no cycle

$$s_1\delta_{i_1} \prec s_2\delta_{i_2} \prec \cdots \prec s_l\delta_{i_l} \prec s_1\delta_{i_1} \tag{3.5}$$

Assume that $\prec$ cannot be extended to a total order on $\{\pm\delta_1, \cdots, \pm\delta_k\}$ compatible with multiplication by -1 and consider a cycle (3.5) of minimal length. It gives rise to a sequence $\nu_1, \cdots, \nu_l \in \mathcal{S}$ induced from (3.4). More precisely,

$$\nu_j = \left\{ \begin{array}{lll} s_j\delta_{i_j} - s_{j+1}\delta_{i_{j+1}} & \text{if} & \delta_{i_j} \neq \delta_{i_{j+1}} \\ s_j\delta_{i_j} & \text{if} & \delta_{i_j} = \delta_{i_{j+1}}, \mathfrak{g} = \mathrm{B}_n \text{ or } \mathfrak{g} = \mathrm{D}_n, \text{ Type II} \\ 2s_j\delta_{i_j} & \text{if} & \delta_{i_j} = \delta_{i_{j+1}}, \mathfrak{g} = \mathrm{C}_n \text{ or } \mathfrak{g} = \mathrm{D}_n, \text{ Type I}, \end{array} \right.$$

where $s_{l+1} = s_1$ and $\delta_{i_{l+1}} = \delta_{i_1}$.

The minimality of (3.5) implies that every element ν of $\mathcal{R}$ appears at most twice in the sequence $\nu_1, \nu_2, \ldots, n_l$. Moreover, if $\nu = \pm\delta_i$ or $\nu = \pm 2\delta_i$, then ν appears at most once in this sequence.

First we consider the case when $\delta_{i_j} \neq \delta_{i_{j+1}}$ for every j. In this case $\nu_j = s_j\delta_{i_j} - s_{j+1}\delta_{i_{j+1}}$ for every j. Let $\lambda_1, \ldots, \lambda_s$ be the elements of $\mathcal{R}$ that appear once in the sequence $\nu_1, \nu_2, \ldots, \nu_l$ and let $\mu_1, \ldots, \mu_t$ be those that appear twice. Clearly, $l = s + 2t$. Moreover $\mathfrak{g}^{\lambda_1} \oplus \cdots \oplus \mathfrak{g}^{\lambda_s} \oplus \mathfrak{g}^{\mu_1} \oplus \cdots \oplus \mathfrak{g}^{\mu_t}$ is a submodule of $\mathcal{M}$ and $\mathrm{Sym}^l(\mathfrak{g}^{\lambda_1} \oplus \cdots \oplus \mathfrak{g}^{\lambda_s} \oplus \mathfrak{g}^{\mu_1} \oplus \cdots \oplus \mathfrak{g}^{\mu_t})$ is a submodule of $\mathrm{Sym}^{\cdot}(\mathcal{M})$ containing

(3.6) $$\mathfrak{g}^{\lambda_1} \otimes \cdots \otimes \mathfrak{g}^{\lambda_s} \otimes \mathrm{Sym}^2 \mathfrak{g}^{\mu_1} \otimes \cdots \otimes \mathrm{Sym}^2 \mathfrak{g}^{\mu_t}.$$

We will prove that the $\mathfrak{s}$-module (3.6) contains the trivial $\mathfrak{s}$-module which, as in the case when $\mathfrak{g} = \mathfrak{gl}_n$, will complete the proof.

Indeed, if $\mu_{j'} = \nu_j$ then

(3.7) $$\begin{aligned} &\mathrm{Sym}^2 \mathfrak{g}^{\mu_{j'}} = \mathrm{Sym}^2 \mathfrak{g}^{\nu_j} = \mathrm{Sym}^2(V_{i_j}^{s_j} \otimes \mathrm{V}_{i_{j+1}}^{-s_{j+1}}) = \\ &\mathrm{Sym}^2 \mathrm{V}_{i_j}^{s_j} \otimes \mathrm{Sym}^2 \mathrm{V}_{i_{j+1}}^{-s_{j+1}} \oplus \Lambda^2 \mathrm{V}_{i_j}^{s_j} \otimes \Lambda^2 \mathrm{V}_{i_{j+1}}^{-s_{j+1}} \supset \mathrm{Sym}^2 \mathrm{V}_{i_j}^{s_j} \otimes \mathrm{Sym}^2 \mathrm{V}_{i_{j+1}}^{-s_{j+1}}. \end{aligned}$$

Replacing in (3.6) each term of the form $\mathrm{Sym}^2 \mathfrak{g}^{\mu_{j'}}$ with the corresponding term $\mathrm{Sym}^2 \mathrm{V}_{i_j}^{s_j} \otimes \mathrm{Sym}^2 \mathrm{V}_{i_{j+1}}^{-s_{j+1}}$ from (3.7), we obtain another submodule of (3.6). This latest submodule is a tensor product of factors of the form $\mathrm{V}_{i_j}^{\pm}$ and $\mathrm{Sym}^2 \mathrm{V}_{i_j}^{\pm}$. Moreover, the component V_i appears in one of the following groups:

$$\mathrm{V}_i^+ \otimes \mathrm{V}_i^+ \otimes \mathrm{V}_i^- \otimes \mathrm{V}_i^-,\ \mathrm{V}_i^+ \otimes \mathrm{V}_i^+ \otimes \mathrm{Sym}^2 \mathrm{V}_i^-,\ \mathrm{V}_i^- \otimes \mathrm{V}_i^- \otimes \mathrm{Sym}^2 \mathrm{V}_i^+,\ \mathrm{Sym}^2 \mathrm{V}_i^+ \otimes \mathrm{Sym}^2 \mathrm{V}_i^-.$$

Since each of them contains the trivial $\mathfrak{s}^i$-module, we conclude that (3.6) contains the trivial $\mathfrak{s}$-module.

Finally, we consider the case when $\delta_{i_j} = \delta_{i_{j+1}}$ for some $1 \leqslant j \leqslant l$. (The minimality of the cycle (3.5) implies that there are at most two such indices but we will not use this observation.) We split the roots $\nu_1, \nu_2, \ldots, \nu_l$ into two groups $\lambda_1, \lambda_2, \ldots, \lambda_s$ and $\mu_1, \mu_2, \ldots, \mu_t$ in the following way: If $\nu_j = s_j\delta_{i_j} - s_{j+1}\delta_{i_{j+1}}$, then we put ν_j in the first or second group depending on whether it appears once or twice in $\nu_1, \nu_2, \ldots, \nu_l$, if $\nu_j = s_j\delta_{i_j}$, we put ν_j in the second group, and if $\nu_j = 2s_j\delta_{i_j}$, we put ν_j in the first group. Set $l' := s + 2t$; note that $l' \neq l$.

From this point on the argument repeats the argument above with the following modifications:

(i) We consider $\mathrm{Sym}^{l'}(\mathfrak{g}^{\lambda_1} \oplus \cdots \oplus \mathfrak{g}^{\lambda_s} \oplus \mathfrak{g}^{\mu_1} \oplus \cdots \oplus \mathfrak{g}^{\mu_t})$ in place of $\mathrm{Sym}^l(\mathfrak{g}^{\lambda_1} \oplus \cdots \oplus \mathfrak{g}^{\lambda_s} \oplus \mathfrak{g}^{\mu_1} \oplus \cdots \oplus \mathfrak{g}^{\mu_t})$.

(ii) In the case when $\mathfrak{g} = D_n$ and $(\mathcal{P}, \sigma)$ is of Type I, we replace $\mathrm{Sym}^2 \mathrm{V}_{i_j}^{s_j} \otimes \mathrm{Sym}^2 \mathrm{V}_{i_{j+1}}^{-s_{j+1}}$ by $\Lambda^2 \mathrm{V}_{i_j}^{s_j} \otimes \Lambda^2 \mathrm{V}_{i_{j+1}}^{-s_{j+1}}$ in (3.7). Correspondingly, V_i appears in one of the following groups

$$\mathrm{V}_i^+ \otimes \mathrm{V}_i^+ \otimes \mathrm{V}_i^- \otimes \mathrm{V}_i^-,\ \mathrm{V}_i^+ \otimes \mathrm{V}_i^+ \otimes \Lambda^2 \mathrm{V}_i^-,\ \mathrm{V}_i^- \otimes \mathrm{V}_i^- \otimes \Lambda^2 \mathrm{V}_i^+,\ \Lambda^2 \mathrm{V}_i^+ \otimes \Lambda^2 \mathrm{V}_i^-.$$

Exactly as above, for $i > 0$, each of the groups above contains the trivial module $\mathfrak{s}^i$-module. Finally, if $\mathfrak{g} = \mathrm{B}_n$ or $\mathfrak{g} = \mathrm{D}_n$ and $(\mathcal{P}, \sigma)$ is of Type II, V_0 appears in groups

$\mathrm{Sym}^2 \mathrm{V}_0$ (one for each $\nu_j = s_j\delta_{i_j}$). Since in these cases $\mathfrak{s}^0 = \mathrm{B}_{|\mathrm{I}_0|}$ or $\mathfrak{s}^0 = \mathrm{D}_{|\mathrm{I}_0|}$, $\mathrm{Sym}^2 \mathrm{V}_0$ contains the trivial $\mathfrak{s}^0$-module. This completes the proof. □

We now turn to the case that $\mathcal{S}$ is not saturated.

3.2. Existence of $\mathfrak{p}_{\mathcal{M}}$ in types A and D. If $\mathfrak{g}$ is of type A there is nothing to prove since every subset $\mathcal{R}$ is saturated and the statement is equivalent to the first part of this section. The situation is the same when $\mathfrak{g} = \mathrm{D}_n$ and $(\mathcal{P}, \sigma)$ is of type I.

Let $\mathfrak{g} = \mathrm{D}_n$ and let $(\mathcal{P}, \sigma)$ be of type II. We will extend the proof of part (*a*) to this case.

First we note that $-2\delta_i \in \mathcal{S}$ and $\delta_i \in \mathcal{S}$ imply that $(\mathrm{Sym}^{\cdot}(\mathcal{M}))^{\mathfrak{s}} \neq \mathbf{C}$. Indeed, $\Lambda^2\mathrm{V}_i^- \oplus \mathrm{V}_i^+ \otimes \mathrm{V}_0$ is a submodule of $\mathcal{M}$ and hence we have the following inclusions of modules:

$$\begin{array}{rcl} \mathrm{Sym}^6(\Lambda^2\mathrm{V}_i^- \oplus \mathrm{V}_i^+ \otimes \mathrm{V}_0) & \subset & \mathrm{Sym}^{\cdot}(\mathcal{M}) \\ \mathrm{Sym}^2(\Lambda^2\mathrm{V}_i^-) \otimes \mathrm{Sym}^4(\mathrm{V}_i^+ \otimes \mathrm{V}_0) & \subset & \mathrm{Sym}^6(\Lambda^2\mathrm{V}_i^- \oplus \mathrm{V}_i^+ \otimes \mathrm{V}_0) \\ \mathrm{S}^{(2,2)}\mathrm{V}_i^+ \otimes \mathrm{S}^{(2,2)}\mathrm{V}_0 \subset \mathrm{Sym}^4(\mathrm{V}_i^+ \otimes \mathrm{V}_0) & , & \mathrm{S}^{(2,2)}\mathrm{V}_i^- \subset \mathrm{Sym}^2(\Lambda^2\mathrm{V}_i^-), \end{array} \tag{3.8}$$

where $\mathrm{S}^{(2,2)}\mathrm{W}$ denotes the result of applying the Schur functor $\mathrm{S}^{(2,2)}$ to W. The above inclusions along the fact that $\mathrm{S}^{(2,2)}\mathrm{V}_0$ contains the trivial $\mathfrak{s}^0$-module imply that $(\mathrm{Sym}^6(\mathcal{M}))^{\mathfrak{s}} \neq 0$. A symmetric argument shows that $2\delta_i \in \mathcal{S}$ and $-\delta_i \in \mathcal{S}$ imply that $(\mathrm{Sym}^{\cdot}(\mathcal{M}))^{\mathfrak{s}} \neq \mathbf{C}$.

From this point on the proof follows the proof of part (*a*) with the following modifications:

(i) In the definition of $\prec$ we use $s_i\delta_i \prec -s_i\delta_i$ if $s_i\delta_i \in \mathcal{S}$ **or** $2s_i\delta_i \in \mathcal{S}$.
(ii) If $s_i\delta_i \prec -s_i\delta_i$, ν_i denotes the corresponding element of $\mathcal{S}$ above; if there are two such elements, we set $\nu_i := s_i\delta_i$.
(iii) In splitting $\nu_1, \nu_2, \ldots, \nu_l$ into two groups $\lambda_1, \lambda_2, \ldots, \lambda_s$ and $\mu_1, \mu_2, \ldots, \mu_t$, we put a root ν_i from (ii) into the first group if $\nu_i = 2s_i\delta_i$ and in the second group otherwise.
(iv) We consider $\mathrm{Sym}^{2l'}(\mathfrak{g}^{\lambda_1} \oplus \cdots \oplus \mathfrak{g}^{\lambda_s} \oplus \mathfrak{g}^{\mu_1} \oplus \cdots \oplus \mathfrak{g}^{\mu_t})$ in place of $\mathrm{Sym}^l(\mathfrak{g}^{\lambda_1} \oplus \cdots \oplus \mathfrak{g}^{\lambda_s} \oplus \mathfrak{g}^{\mu_1} \oplus \cdots \oplus \mathfrak{g}^{\mu_t})$.
(v) We replace the module in (3.6) by $\mathrm{Sym}^2 \mathfrak{g}^{\lambda_1} \otimes \cdots \otimes \mathrm{Sym}^2 \mathfrak{g}^{\lambda_s} \otimes \mathrm{Sym}^4 \mathfrak{g}^{\mu_1} \otimes \cdots \otimes \mathrm{Sym}^4 \mathfrak{g}^{\mu_t}$.

Using the inclusions (3.8) we conclude that $(\mathrm{Sym}^{\cdot}(\mathcal{M}))^{\mathfrak{s}} \neq \mathbf{C}$. This completes the proof when $\mathfrak{g} = \mathrm{D}_n$. □

3.3. Examples in types B and C when $\mathcal{M}$ is not saturated. We will now construct examples in types B and C of $\mathfrak{s}$ and $\mathcal{S}$ such that $(\mathrm{Sym}^{\cdot}(\mathcal{M}))^{\mathfrak{s}} = \mathbf{C}$ and for which there does not exist a parabolic subalgebra $\mathfrak{p}_{\mathcal{M}}$ of $\mathfrak{g}$ with reductive part $\mathfrak{s}$ and $\mathcal{M} \subset \mathfrak{p}_{\mathcal{M}}$.

If $\mathfrak{g} = \mathrm{B}_n$, consider $\mathfrak{s} = \mathfrak{s}_{(\mathcal{P},\sigma)}$, where $\mathcal{P}$ is the partition of type I

$$\{1,2\} \prec \{3\} \prec \{4\} \prec \cdots \prec \{n\}$$

and $\sigma(i) = 1$ is constant. Then $\mathfrak{s}^1 = \mathfrak{gl}_2$. Moreover, $\mathrm{U} := \mathfrak{g}^{-\delta_1}$ is the $\mathfrak{gl}_2$-module which is the natural representation of $\mathfrak{sl}_2$ and on which the identity matrix of $\mathfrak{gl}_2$ acts as multiplication by -1 and $\mathrm{W} := \mathfrak{g}^{2\delta_1}$ is the one dimensional $\mathfrak{gl}_2$-module on which the identity matrix acts as multiplication by 2. Let $\mathcal{S} := \{-\delta_1, 2\delta_1\}$. Then $\mathcal{M} = \mathrm{U} \oplus \mathrm{W}$ and

$$\mathrm{Sym}^k \mathcal{M} = \oplus_j \mathrm{Sym}^j \mathrm{U} \otimes \mathrm{Sym}^{k-j} \mathrm{W}.$$

Note that $\mathrm{Sym}^j \mathrm{U} \otimes \mathrm{Sym}^{k-j} \mathrm{W}$ is the irreducible $\mathfrak{sl}_2$-module of dimension $j+1$ on which the identity matrix of $\mathfrak{gl}_2$ acts as multiplication by $2k - 3j$. This proves that $(\mathrm{Sym}^{\cdot}(\mathcal{M}))^{\mathfrak{s}} = \mathbf{C}$ but there is no parabolic subalgebra $\mathfrak{p}_{\mathcal{M}}$ of $\mathfrak{g}$ with reductive part $\mathfrak{s}$ such that $\mathcal{M} \subset \mathfrak{p}_{\mathcal{M}}$.

If $\mathfrak{g} = \mathrm{C}_n$, consider $\mathfrak{s} = \mathfrak{s}_{(\mathcal{P},\sigma)}$, where $\mathcal{P}$ is the partition of type II

$$\{1\} \prec \{2\} \prec \{3\} \prec \cdots \prec \{n\}$$

and $\sigma(i) = 1$ is constant. Then $\mathfrak{s}^0 = \mathrm{C}_1 \cong \mathfrak{sl}_2$ and $\mathfrak{s}^1 = \mathfrak{gl}_1$, i.e. $\mathfrak{s}^0 \oplus \mathfrak{s}^1 \cong \mathfrak{gl}_2$. Moreover, setting $\mathrm{U} := \mathfrak{g}^{-\delta_1}$ and $\mathrm{W} := \mathfrak{g}^{2\delta_1}$, we arrive at exactly the same situation as in the case $\mathfrak{g} = \mathrm{B}_n$ above. □

4. PROOF OF THE MAIN THEOREM WHEN $\mathfrak{g}$ IS EXCEPTIONAL.

4.1. First we recall some standard notation following the conventions in [B]. If $\mathfrak{g}$ is a simple Lie algebra of rank n we label the simple roots of $\mathfrak{g}$ as $\alpha_1, \ldots, \alpha_n$ as in [B]. The fundamental dominant weight of $\mathfrak{g}$ are denoted by $\omega_1, \ldots, \omega_n$. If $-\alpha_0$ is the highest root, then $\alpha_0, \alpha_1, \ldots, \alpha_n$ label the extended Dynkin diagram of $\mathfrak{g}$.

4.2. Existence of $\mathfrak{p}_{\mathcal{M}}$ in type G_2 when $\mathcal{S}$ is saturated. Let $\mathfrak{g} = \mathrm{G}_2$. Let $\mathcal{S}$ be a saturated subset of $\mathcal{R}$ and let $\mathcal{M} = \oplus_{\nu \in \mathcal{S}} \mathfrak{g}^\nu$. If $(\mathrm{Sym}^{\cdot}(\mathcal{M}))^{\mathfrak{s}} = \mathbf{C}$, then there exists a parabolic subalgebra $\mathfrak{p}_{\mathcal{M}}$ of $\mathfrak{g}$ with reductive part $\mathfrak{s}$ such that $\mathcal{M} \subset \mathfrak{p}_{\mathcal{M}}$. Indeed, if $\mathfrak{s}$ is a proper subalgebra of $\mathfrak{g}$ which is not equal to $\mathfrak{h}$, then all elements of $\mathcal{R}$ are proportional and there is nothing to prove. If $\mathfrak{s} = \mathfrak{h}$, then the spaces $\mathfrak{g}^\nu$ are just the root spaces of $\mathfrak{g}$ which are one dimensional and again the statement is clear. □

4.3. Example in type G_2 when $\mathcal{S}$ is not saturated. On the other hand, let $\mathfrak{s} \cong \mathfrak{gl}_2 \subset \mathfrak{g}$ be the parabolic subalgebra of $\mathfrak{g}$ with roots $\pm\alpha_2$. Then $\mathcal{R} = \{\pm\delta, \pm 2\delta, \pm 3\delta\}$. Moreover, $\mathfrak{g}^{\pm k\delta}$ is the irreducible $\mathfrak{s}$-module of dimension $2, 1$, or 2 (corresponding to $k = 1, 2$, or 3) on which a fixed element in the centre of $\mathfrak{s}$ acts as multiplication by $\pm k$. Then, for $\mathcal{S} = \{-\delta, 2\delta\}$, setting $\mathrm{U} := \mathfrak{g}^{-\delta}$ and $\mathrm{W} := \mathfrak{g}^{2\delta}$, we arrive at exactly the same situation as at the end of Section 2 above. In particular, $(\mathrm{Sym}^{\cdot}(\mathcal{M}))^{\mathfrak{s}} = \mathbf{C}$ but there is no parabolic subalgebra $\mathfrak{p}_{\mathcal{M}}$ of $\mathfrak{g}$ with reductive part $\mathfrak{s}$ such that $\mathcal{M} \subset \mathfrak{p}_{\mathcal{M}}$. □

4.4. Examples in types F_4, E_6, E_7, and E_8 with $\mathcal{S}$ saturated. Let $\mathfrak{g} = \mathrm{F}_4, \mathrm{E}_6, \mathrm{E}_7$, or E_8. We will construct a saturated set $\mathcal{S}$ such that $(\mathrm{Sym}^{\cdot}(\mathcal{M}))^{\mathfrak{s}} = \mathbf{C}$ but there is no parabolic subalgebra $\mathfrak{p}_{\mathcal{M}}$ of $\mathfrak{g}$ with reductive part $\mathfrak{s}$ such that $\mathcal{M} \subset \mathfrak{p}_{\mathcal{M}}$.

Denote the rank of $\mathfrak{g}$ by n. Consider the extended Dynkin diagram of $\mathfrak{g}$. Removing the node connected to the root α_0 we obtain the Dynkin diagram of a semisimple subalgebra $\mathfrak{m} \oplus \mathfrak{c}$ of $\mathfrak{g}$ of rank n where $\mathfrak{m} \cong \mathrm{A}_1$ is the subalgebra of $\mathfrak{g}$ with roots $\{\pm\alpha_0\}$ and $\mathfrak{c}$ is the subalgebra if $\mathfrak{g}$ with simple roots obtained from the simple roots of $\mathfrak{g}$ after removing the one adjacent to α_0. More precisely, we remove the roots $\alpha_1, \alpha_2, \alpha_1, \alpha_8$ when $\mathfrak{g} = \mathrm{F}_4, \mathrm{E}_6, \mathrm{E}_7, \mathrm{E}_8$ respectively. The respective subalgebras $\mathfrak{c} \subset \mathfrak{g}$ are isomorphic to $\mathfrak{c} \cong \mathrm{C}_3, \mathrm{A}_5, \mathrm{D}_6$, or E_7 respectively. As an $\mathfrak{m}$–module $\mathfrak{g}$ decomposes as

$$\mathfrak{g} = (\mathrm{Ad}_{\mathfrak{m}} \otimes \mathrm{tr}_{\mathfrak{c}}) \oplus (\mathrm{tr}_{\mathfrak{m}} \otimes \mathrm{Ad}_{\mathfrak{c}}) \oplus (\mathrm{V} \otimes \mathrm{U}), \tag{4.1}$$

where $\mathrm{Ad}_{\mathfrak{m}}$ and $\mathrm{Ad}_{\mathfrak{c}}$ are the adjoint modules of $\mathfrak{m}$ and $\mathfrak{c}$ respectively; $\mathrm{tr}_{\mathfrak{m}}$ and $\mathrm{tr}_{\mathfrak{c}}$ —the respective trivial modules; V is the natural $\mathfrak{m} \cong \mathrm{A}_1$–module; and U is the $\mathfrak{c}$–module whose highest weight is the fundamental weight of $\mathfrak{c}$ corresponding to the simple root of $\mathfrak{c}$ linked to the removed node of the extended Dynkin diagram of $\mathfrak{g}$. In fact, for $\mathfrak{g} = \mathrm{F}_4, \mathrm{E}_6, \mathrm{E}_7, \mathrm{E}_8$, the highest weight of $\mathfrak{c}$ is $\omega_3, \omega_3, \omega_6, \omega_7$ respectively. Here the weights of U are given according to the labeling conventions of $\mathfrak{c}$. For example, if $\beta_1, \beta_2, \beta_3$ are the simple roots of $\mathfrak{c} = \mathrm{C}_3$ in the case when $\mathfrak{g} = \mathrm{F}_4$, we have $\beta_1 = \alpha_4$, $\beta_2 = \alpha_3$, and $\beta_3 = \alpha_2$.

Set $\mathfrak{s} = \mathfrak{m} + \mathfrak{h}$. From the construction of $\mathfrak{s}$ we conclude that $\mathfrak{t} = \mathfrak{h}_\mathfrak{c}$, the Cartan subalgebra of $\mathfrak{c}$. Furthermore, (4.1) implies $\mathcal{R} = \Delta_\mathfrak{c} \cup \operatorname{supp} \mathrm{U}$, where $\operatorname{supp} \mathrm{U}$ denotes the set of weights of U and, for $\nu \in \mathcal{R}$ the $\mathfrak{s} = \mathfrak{m} \oplus \mathfrak{h}_\mathfrak{c}$–module $\mathfrak{g}^\nu$ is given by

$$\mathfrak{g}^\nu \cong \begin{cases} \mathrm{tr}_\mathfrak{m} \otimes \nu & \text{if } \nu \in \Delta_\mathfrak{c} \\ \mathrm{V} \otimes \nu & \text{if } \nu \in \operatorname{supp} \mathrm{U}. \end{cases}$$

Let ω be the highest weight of U and write $\omega = q_1\beta_1 + \cdots + q_{n-1}\beta_{n-1}$ where $\beta_1, \ldots, \beta_{n-1}$ are the simple roots of $\mathfrak{c}$ and $q_i \in \mathbf{Q}_+$. Set $\mathcal{S} = \{-\omega, \beta_1, \ldots, \beta_{n-1}\}$. Then $\mathcal{M} = \mathfrak{g}^\omega \oplus (\oplus_{i=1}^{n-1} \mathfrak{g}^{\beta_i})$ and

$$\operatorname{Sym}^k \mathcal{M} = \bigoplus_{j+i_1+\cdots+i_{n-1}=k} \operatorname{Sym}^j \mathfrak{g}^{-\omega} \otimes \operatorname{Sym}^{i_1} \mathfrak{g}^{\beta_1} \otimes \cdots \otimes \operatorname{Sym}^{i_{n-1}} \mathfrak{g}^{\beta_{n-1}}.$$

Moreover, $\operatorname{Sym}^j \mathfrak{g}^{-\omega} \otimes \operatorname{Sym}^{i_1} \mathfrak{g}^{\beta_1} \otimes \cdots \otimes \operatorname{Sym}^{i_{n-1}} \mathfrak{g}^{\beta_{n-1}}$ is an irreducible $\mathfrak{m}$–module which is not trivial unless $j = 0$ and on which $\mathfrak{h}_\mathfrak{c}$ acts via $-j\omega + i_1\beta_1 + \cdots + i_{n-1}\beta_{n_1}$. This implies that, for $k > 0$, $(\operatorname{Sym}^k \mathcal{M})^\mathfrak{s} = 0$ and hence $(\operatorname{Sym}^\cdot \mathcal{M})^\mathfrak{s} = \mathbf{C}$. On the other hand, the equation $\omega = q_1\beta_1 + \cdots + q_{n-1}\beta_{n-1}$ implies that there is no parabolic subalgebra $\mathfrak{p}_\mathcal{M}$ of $\mathfrak{g}$ with reductive part $\mathfrak{s}$ such that $\mathcal{M} \subset \mathfrak{p}_\mathcal{M}$. □

REFERENCES

[B] N. Bourbaki, *Éléments de mathématique. Groupes et algèbres de Lie*, Ch. IV – VI, Herman, Paris 1968, 288 pp.

[DFG] I. Dimitrov, V. Futorny, and D. Grantcharov, *Parabolic sets of roots*, Contemp. Math. **499** (2009), 61–74.

[DP] I. Dimitrov and I. Penkov, *Weight modules of direct limit Lie algebras*, IMRN 1999, No. 5, 223–249.

[DR] I. Dimitrov, M. Roth, *Cup products of line bundles on homogeneous varieties and generalized PRV components of multiplicity one*, to appear in Algebra & Number Theory.

[K] B. Kostant, *Root systems for Levi factors and Borel–de Siebenthal theory*, *Symmetry and Spaces*, 129–152, Progr. Math., 278, Birkhäuser Boston, Inc., Boston, MA, 2010.

DEPARTMENT OF MATHEMATICS AND STATISTICS, QUEEN'S UNIVERSITY, KINGSTON, ONTARIO, K7L 3N6, CANADA

E-mail address: dimitrov@mast.queensu.ca

E-mail address: mikeroth@mast.queensu.ca