ABEL-JACOBI MAPS ASSOCIATED TO SMOOTH CUBIC THREEFOLDS

JOE HARRIS, MIKE ROTH, AND JASON STARR

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Abstract. In this article we consider the spaces $\mathcal{H}^{d,g}(X)$ parametrizing curves of degree d and genus g on a smooth cubic threefold $X \subset \mathbb{P}^4$, with regard in particular to the Abel-Jacobi map $u_d : \mathcal{H}^{d,g}(X) \to J^3(X)$ to the intermediate Jacobian $J^3(X)$ of X. Our principle result is that for all $d \leq 5$ the map u_d coincides with the maximal rationally connected fibration of $\mathcal{H}^{d,g}(X)$.

1. Introduction

In this paper we will study spaces parametrizing curves on a smooth cubic threefold $X \subset \mathbb{P}^4$. There has been a great deal of work in recent years on the geometry of spaces parametrizing rational curves on a variety X. Most of it has focussed on the enumerative geometry of these spaces: the description of their Chow rings and the evaluation of certain products in their Chow rings. Here we will be concerned with a very different sort of question: we will be concerned with the birational geometry of the spaces M.

We should start by explaining, at least in part, our motivation. The central object in curve theory—the one that links together every aspect of the theory, and whose study yields the majority of theorems in the subject—is the Abel-Jacobi map. This is the map from the symmetric product C_d of a curve C, parametrizing 0-cycles of a given degree d on C, to the Jacobian $\mathrm{Pic}^d(C) \cong J(C)$, defined variously as the space of cycles of degree d mod linear equivalence (that is, the space of line bundles of degree d on C) or, over the complex numbers, as the complex torus $H^1(C, K)^*/H_1(C, \mathbb{Z})$.

Under the circumstances, it's natural to ask what sort of analogue of this we might be able to define and study in higher dimensions. There have been numerous constructions proposed to this end; they have been too varied to categorize easily, but the majority adopt one of two approaches, corresponding to the definitions of $\mathrm{Pic}^d(C) \cong J(C)$ for a curve. In the first, we look again at the space of cycles on a variety X mod some equivalence relation (typically rational equivalence); in the second, we try to form a geometric object out of the Hodge structure of X.

Both approaches suffer from some seemingly unavoidable difficulties. In the first, the quotients of the spaces on cycles on a variety X by rational equivalence tends to be either too big or too small: if we mod out by algebraic equivalence we tend to lose too much information, while if we mod out by rational equivalence the quotient is too large (and in particular too hard to calculate even in simple concrete cases). As for the

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second, we simply don't know how to make an algebrao-geometric object out of a Hodge structure except in very special cases—specifically, the case of Hodge structures of odd weight with all but two of the Hodge groups vanishing, where we can form the intermediate Jacobian and it will be an Abelian variety. In these cases the theory has had spectacular successes (such as the proof of the irrationality of cubic threefolds), but it's undeniably frustrating that it has no wider application.

In this paper we'd like to suggest another approach to the problem, based on a construction of Kollár, Miyaoka and Mori: the *maximal rationally connected fibration* associated to a variety X. Briefly, we say that a smooth projective variety X is *rationally connected* if two general points of X can be connected by a chain of rational curves. This condition is equivalent to rationality and unirationality for curves and surfaces; in higher dimensions it is weaker than either. More generally, the maximal rationally connected fibration associates to a variety X a (birational isomorphism class of) variety Z and a rational map $\phi : X \to Z$ with the properties that

- the fibers X_z of ϕ are rationally connected; and conversely
- almost all the rational curves in X lie in fibers of ϕ: for a very general point $z \in Z$ any rational curve in X meeting X_z lies in X_z.

The variety Z and morphism ϕ are unique up to birational isomorphism, and are called the *mrc quotient* and *mrc fibration* of X, respectively. They measure the failure of X to be rationally connected: if X is rationally connected, Z is a point, while if X is not uniruled we have $Z = X$.

The point of this is, if we start with the symmetric product C_d of a curve C and apply this construction, we find (for d large, at any rate) that the mrc fibration of C_d is the Abel-Jacobi map $\phi : C_d \to J(C)$. In other words, we don't have to define the Jacobian $J(C)$ either in terms of cycles mod linear equivalence or via Hodge theory; we can realize it simply the mrc quotient of the space C_d parametrizing 0-cycles of reasonably large degree d on C. The obvious question is then: what happens when we take the mrc quotient of the variety parametrizing cycles[1] on a higher-dimensional variety?

To begin with, taking the mrc quotient of the varieties parametrizing 0-cycles on X doesn't seem to yield much: if X is rationally connected, so will be its symmetric powers; and if X possesses any holomorphic forms the dimensions of the mrc quotients of X_d tends to ∞ with d. (** reference; and also can we say more: for example, if X is not uniruled, can its symmetric powers be?) We turn our attention next to curves on X and ask: what can we say about the mrc quotients of the varieties $\mathcal{H}^{d,g}(X)$ parametrizing curves of degree d and genus g on X? For various reasons it makes sense to look primarily at the spaces parametrizing rational curves—for one thing, we don't want to get involved in the geometry of the moduli space M_g of curves of genus g when we are only interested in invariants of X—but in the course of studying rational curves we will also discover facts about the geometry of $\mathcal{H}^{d,g}(X)$ for $g > 0$.

For $X = \mathbb{P}^n$ and for $X \subset \mathbb{P}^{n+1}$ a quadric hypersurface, the answer is known, at least in the case of rational curves, and is trivial: the variety $\mathcal{H}^{d,0}(X)$ parametrizing rational curves of degree d on X is itself rationally connected in both cases. The first real test is thus cubic hypersurfaces, and here we will focus specifically on the case of a smooth cubic threefold $X \subset \mathbb{P}^4$.

We start by asking the most naive possible question: is the variety $\mathcal{H}^{d,g}(X)$ rationally connected? The answer is that it can never be: we have the Abel Jacobi map

$$u : \mathcal{H}^{d,g}(X) \to J^3(X)$$

from the Kontsevich space to the intermediate Jacobian

$$J_3(X) = H^{2,1}(X)^*/H_3(X, \mathbb{Z}).$$

Since any map of a rational curve to a complex torus/Abelian variety is constant, any rational curve on $\mathcal{H}^{d,g}(X)$ must lie in a fiber of u. The Abel-Jacobi map thus factors through the mrc fibration $\phi : \mathcal{H}^{d,g}(X) \to Z$ of $\mathcal{H}^{d,g}(X)$.

[1]Note that we have a choice of parameter spaces for the cycles on X: the Chow variety, the Hilbert scheme and, in the case of 1-dimensional cycles, the Kontsevich space. But since we are concerned with the birational geometry of these spaces it really doesn't matter which we choose to work with, except for technical convenience.

Given this, the second most naive question would be: is the intermediate Jacobian $J^3(X)$ the mrc quotient of $\mathcal{H}^{d,g}(X)$? This may seem equally naive, but it turns out to hold for small values of d. The main result of this paper is the

Theorem 1.1. *If $X \subset \mathbb{P}^4$ is any smooth cubic threefold, for any $d \leq 5$ the Abel-Jacobi map*

$$u : \mathcal{H}^{d,g}(X) \to J^3(X)$$

is the maximal rationally connected fibration of $\mathcal{H}^{d,g}(X)$.

As we indicated, we view this result as simply the first test of a general hypothesis. It would be very interesting to see if the analogous statement holds for other rationally connected threefolds. Work is being done in this direction: Ana-Maria Castravet has proved in [C] that for $X \subset \mathbb{P}^5$ the intersection of two quadrics, the analogous statement—that the mrc quotient of the space $\mathcal{H}^{d,0}(X)$ is the intermediate Jacobian—holds for all d. And work of Harris and Starr ([HS]) shows that the same is true at least for components of the variety $\mathcal{H}^{d,0}(X)$ corresponding to an open cone of curve classes $\beta \in H_2(Z, \mathbb{Z})$ in a blow-up X of $\mathbb{P}^3$.

Ultimately, though, we want to extend this research to higher-dimensional varieties X, where the Hodge theory doesn't seem to provide us with a candidate for the mrc quotients of the spaces of rational curves on X. To relate this to a specific issue: for smooth cubic threefolds X, the fact that the Hodge structure on $H^3(X, \mathbb{Z})$ is not isomorphic to a product of Hodge structures of curves implies that X is not rational. A close examination of the Hodge structure of a very general cubic fourfold $X \subset \mathbb{P}^5$ suggests a similar finding: the Hodge structure on $H^4(X, \mathbb{Z})$ does not appear to be a product of factors of Hodge structures of surfaces (see [Has]), and if this is indeed the case it would imply that X is irrational. But the fact that the Hodge structure is not conveyed in the form of a geometric object—there is no construction analogous to that of the intermediate Jacobian of a cubic threefold—has frustrated our attempts to make this into a proof. But now we can ask: if the intermediate Jacobian of a cubic threefold X may be realized as the mrc quotient of the space of rational curves on X, what happens when we take the mrc quotient of the space of rational curves on a cubic fourfold?

In this paper, though, we will be concerned exclusively with the geometry of cubic threefolds. We start some preliminary sections on the Abel-Jacobi map for curves on threefolds in general and on the geometry of curves of low degree on cubic threefolds in particular. We then launch into the analysis of the Abel-Jacobi map for curves of degree 3, 4 and 5 on a smooth cubic threefold X. The basic technique here is *residuation*: we associate to a curve $C \subset X$ on X and a surface $S \subset \mathbb{P}^4$ containing it, a residual curve C' with (roughly) $C + C' = S \cap X$. In this way we relate the parameter space for curves C to that for curves C'. In [6] it is proved that for $d \leq 5$, each of the spaces $\mathcal{H}^{d,g}(X)$ is irreducible of dimension $2d$. Combining this with the residuation technique, we prove that for $d \leq 5$ the fibers of the Abel-Jacobi map $\mathcal{H}^{d,g}(X) \to J^3(X)$ are unirational.

Many of the results in this paper have also been proved in [10] and [9] by considering moduli of vector bundles on X.

1.1. **Notation.** All schemes in this paper will be schemes over $\mathbb{C}$. All absolute products will be understood to be fiber products over $\mathrm{Spec}(\mathbb{C})$.

For a projective variety X and a numerical polynomial $P(t)$, $\mathrm{Hilb}_{P(t)}(X)$ denotes the corresponding Hilbert scheme. For integers d, g, $\mathcal{H}^{d,g}(X) \subset \mathrm{Hilb}_{dt+1-g}(X)$ denotes the open subscheme parametrizing smooth, connected curves of degree d and genus g, and $\overline{\mathcal{H}^{d,g}}(X)$ denotes the closure of $\mathcal{H}^{d,g}(X)$ in $\mathrm{Hilb}_{dt+1-g}(X)$.

2. Review of the Abel-Jacobi Map

Our object of study are the *Abel-Jacobi maps* associated to families of 1-cycles on a smooth cubic hypersurface $X \subset \mathbb{P}^4$. The reader is referred to [1] and [5] for full definitions. Here we recall only a few facts about Abel-Jacobi maps.

Associated to a smooth, projective threefold X there is a complex torus

$$J^2(X) = H^3_{\mathbb{Z}}(X) \backslash H^3(X, \mathbb{C}) / \left(H^{3,0}(X) \oplus H^{2,1}(X)\right). \tag{1}$$

In case X is a cubic hypersurface in $\mathbb{P}^4$ (in fact for any *rationally connected* threefold) then $J^2(X)$ is a principally polarized abelian variety with theta divisor Θ. Given an algebraic 1-cycle $\gamma \in A_1(X)$ which is *homologically equivalent to zero* [5, 13], one can associate a point $u_2(\alpha)$. The construction is analogous to the Abel-Jacobi map for a smooth, projective algebraic curve C which associates to each 0-cycle $\gamma \in A_0(C)$ which is homologically equivalent to zero a point $u_1(\alpha) \in J^1(C)$, the Jacobian variety of C. In particular $u_2 : A_1(X)^{hom} \to J^2(X)$ is a group homomorphism.

Suppose that B is a normal, connected variety of dimension n and $\Gamma \in A_{n+1}(B\times X)$ is an $(n+1)$-cycle such that for each closed point $b \in B$ the corresponding cycle $\Gamma_b \in A_1(X)$ [3, §10.1] is homologically equivalent to zero. Then in this case the set map $b \mapsto u_1(\Gamma_b) \in J^2(X)$ comes from a (unique) algebraic morphism $u = u_\Gamma : B \to J^2(X)$. We call this morphism the *Abel-Jacobi map* determined by Γ.

More generally, suppose B as above, $\Gamma \in A_{n+1}(B \times X)$ is any $(n+1)$-cycle, and suppose $b_0 \in B$ is some base-point. Then we can form a new cycle $\Gamma' = \Gamma - \pi_2^*\Gamma_{b_0}$, and for all $b \in B$ we have $\Gamma'_b = \Gamma_b - \Gamma_{b_0}$ is homologically equivalent to zero. Thus we have an algebraic morphism $u = u_{\Gamma'} : B \to J^2(X)$. Of course this morphism depends on the choice of a base-point, but changing the base-point only changes the morphism by a constant translation. Thus we shall speak of any of the morphisms $u_{\Gamma'}$ determined by Γ and the choice of a base-point as an *Abel-Jacobi map* determined by Γ.

Suppose that $\Gamma_1, \Gamma_2 \in A_{n+1}(B \times X)$ are two $(n+1)$-cycles. Then $u_{\Gamma_1+\Gamma_2}$ is the pointwise sum $u_{\Gamma_1} + u_{\Gamma_2}$. This trivial observation is frequently useful.

The Residuation Trick: Another useful observation is that any Abel-Jacobi morphism α_Γ contracts all rational curves on X, since an Abelian variety contains no rational curves. Combined with the observation in the last paragraph, this leads to the residuation trick: Suppose that B is a normal, unirational variety and $\Gamma \in A_{n+1}(B \times X)$ is an $(n+1)$-cycle. Then $u_\Gamma : B \to J(X)$ is a constant map. Now suppose that $B' \subset B$ is a normal closed subvariety and that $\Gamma|_{B'\times X}$ decomposes as a sum of cycles $\Gamma_1 + \Gamma_2$. Since $u_{\Gamma_1} + u_{\Gamma_2}$ equals a constant map, we conclude that u_{Γ_1} is the pointwise inverse of u_{Γ_2}, up to a fixed additive constant.

3. Lines, Conics and Plane Cubics

The study of the Abel-Jacobi map associated to the space $\mathcal{H}^{1,0}(X)$ of lines on X was carried out in [1]. In this section we will summarize their results, which will also be useful to us for studying Abel-Jacobi maps of higher degree curves. In this section we also consider the Abel-Jacobi maps associated to the spaces $\mathcal{H}^{2,0}(X)$ and $\mathcal{H}^{3,1}(X)$ of plane conics and plane cubics in X. In each case the Abel-Jacobi is trivial to describe.

3.1. **Lines.** For brevity we refer to the *Fano scheme of lines*, $\mathcal{H}^{1,0}(X)$, simply as F. Two general lines $L_1, L_2 \subset \mathbb{P}^4$ determine a hyperplane by $\mathrm{span}(L_1, L_2)$. We generalize this as follows: Let $(F \times F - \Delta) \xrightarrow{\Phi} \mathbb{P}^{4\vee}$ denote the following set map:

$$\Phi\left([L_1, L_2]\right) = \begin{cases} [\mathrm{span}\,(L_1, L_2)] & \text{if } L_1 \cap L_2 = \emptyset, \\ [T_pX] & \text{if } p \in L_1 \cap L_2 \end{cases} \tag{2}$$

By [1, lemma 12.16], Φ is algebraic. Let $X^\vee \subset \mathbb{P}^{4\vee}$ denote the dual variety of X, i.e. the variety parametrizing tangent hyperplanes to X. Let $X_s^\vee \subset X^\vee$ denote the subvariety parametrizing hyperplanes H which are tangent to X and such that the singular locus of $H \cap X$ is not simply a single ordinary double point. Let $U_s \subset U \subset F \times F$ denote the open sets $\Phi^{-1}\left(\mathbb{P}^{4\vee} - X^\vee\right) \subset \Phi^{-1}\left(\mathbb{P}^{4\vee} - X_s^\vee\right)$. And let $I \subset F \times F$ denote the divisor parametrizing incident lines, i.e. I is the closure of the set $\{([L_1], [L_2]) : L_1 \neq L_2, L_1 \cap L_2 \neq \emptyset\}$. In [1], Clemens and Griffiths completely describe both the total Abel-Jacobi map $F \times F \xrightarrow{\psi} J(X)$ and the Abel-Jacobi map $F \xrightarrow{i} J(X)$. Here is a summary of their results

Theorem 3.1. (1) *The Fano variety F is a smooth surface and the Abel-Jacobi map $F \xrightarrow{u} J(X)$ is a closed immersion* [1, theorem 7.8, theorem 12.37].

(2) *The induced map $Alb(F) = J^2(F) \to J(X)$ is an isomorphism of principally polarized Abelian varieties* [1, theorem 11.19].

(3) *The class of $u(F)$ in $J(X)$ is $\frac{[\Theta]^3}{3!}$* [1, proposition 13.1].

(4) *The difference of Abel-Jacobi maps*

$$\psi : F \times F \to J(X), \quad \psi([L],[L']) = u([L]) - u([L']) \tag{3}$$

maps $F \times F$ generically 6*-to-*1 *to the theta divisor $\Theta \subset J(X)$* [1, section 13].

(5) *Let $(\Theta - \{0\}) \xrightarrow{\mathcal{G}} \mathbb{P}H^{1,2}(X)^\vee$ denote the Gauss map. If we identify $\mathbb{P}H^{1,2}(X)$ with $\mathbb{P}^4$ via the Griffiths residue calculus* [4]*, then the composite map*

$$(F \times_C F - \Delta) \xrightarrow{\psi} (\Theta - \{0\}) \xrightarrow{\mathcal{G}} \mathbb{P}^{4\vee} \tag{4}$$

is just the map Φ defined above [1, formula 13.6].

(6) *The fibers of the Abel-Jacobi map form a* Schläfli double-six*, i.e. the general fiber of $\psi : F \times F \to J$ is of the form $\{(E_1, G_1), \ldots, (E_6, G_6)\}$ where the lines E_i, G_j lie in a smooth hyperplane section of X, the E_i are pairwise skew, the G_j are pairwise skew, and E_i and G_j are skew iff $i = j$.*

There is a more precise result than above. Let

$$R' \subset (U \times_{\mathbb{P}^{4\vee}} U) \times F \times Grass(3,V) \times Grass(3,V) \tag{5}$$

be the closed subscheme parametrizing data $(([L_1],[L_2]), ([L_3],[L_4]), [l], [H_1], [H_3])$ such that for each $i = 1, \ldots, 4$, $l \cap L_i \neq \emptyset$ and such that $H_1 \cap X = l \cup L_1 \cup L_4$, $H_2 \cap X = l \cup L_2 \cup L_3$. Let $R \subset U \times_{\mathbb{P}^{4\vee}} U$ be the image of R' under the projection map. Let $\Delta \subset U \times U$ be the diagonal. Then the fiber product $U \times_\Theta U \subset U \times U$ is just the union $R \cup \Delta$ [1, p. 347-348]

(7) *The branch locus of $\Theta \xrightarrow{\mathcal{G}} \mathbb{P}^{4\vee}$ equals the branch locus of $F \times F \xrightarrow{\Phi} \mathbb{P}^{4\vee}$ equals the dual variety of X, i.e. the variety parametrizing the tangent hyperplanes to X. The ramification locus of $U \xrightarrow{\Phi} \mathbb{P}^{4\vee}$ equals the ramification locus of $U \xrightarrow{\psi} \Theta$ equals the divisor I. Each such pair is a simple ramification point of both ψ and Φ* [1, lemma 13,8].

In [12], Tjurin also analyzed cubic threefolds and the associated intermediate Jacobians. We summarize his results:

Theorem 3.2. [12] *Let $\widetilde{J(X)}$ be the variety obtained by blowing up $0 \in J(X)$ and let $\widetilde{\Theta}$ be the proper transform of Θ. Let $\widetilde{F \times F}$ be the variety obtained by blowing up the diagonal in $F \times F$. The exceptional divisor $E \subset \widetilde{J(X)}$ is isomorphic to $\mathbb{P}^4$ and this isomorphism identifies the intersection $\widetilde{\Theta} \cap E$ with our original cubic threefold X. The exceptional divisor $E' \subset \widetilde{F \times F}$ is the projective bundle $\mathbb{P}T_F$. In fact there is an isomorphism of sheaves of the tautological rank 2 sheaf $S = S(2,V)$ and the tangent bundle T_F so that $E' \cong \mathbb{P}_F S$. The induced morphism $\mathbb{P}_F S \xrightarrow{\tilde{\psi}} \mathbb{P}^4$ is just the usual morphism induced by the map of sheaves $S \hookrightarrow V \otimes_{\mathbb{C}} \mathcal{O}_F$. In particular, this morphism is generically* 6*-to-*1 *onto X with ramification locus $\Sigma_1 \subset F$ consisting of the lines of "second type", i.e. lines $L \subset X$ such that $N_{L/X} \cong \mathcal{O}_L(-1) \oplus \mathcal{O}_L(1)$. Let $D \subset F \times F$ denote the divisor of intersecting lines and let $\widetilde{D}$ be the proper transform of D. Then $\widetilde{D} \cap E' = \Sigma_1$ so that we finally have the result: The morphism $\widetilde{F \times F} \to \widetilde{\Theta}$ is finite of degree* 6 *with ramification locus $\widetilde{D}$.*

3.2. **Conics.** Next we consider the space $\mathcal{H}^{2,0}(X)$ parametrizing smooth plane conics on X. Given a plane conic $C \subset X$, consider the plane cubic curve which is the intersection $\mathrm{span}(C) \cap X$. This curve contains C is an irreducible component, and the residual component is a line $L \subset X$. In this way we have a morphism $\mathcal{H}^{2,0}(X) \to F$ by $[C] \mapsto [L]$. Conversely, given a line $L \subset X$ and a 2-plane $P \subset \mathbb{P}^4$ which contains L, then the residual to L in $P \cap X$ is a plane conic. In this way one sees that $\mathcal{H}^{2,0}(X)$ is isomorphic to an open subset of the $\mathbb{P}^2$-bundle

$$\mathbb{P}Q = \{([L],[P]) \in F \times \mathbf{G}(2,4) | L \subset P\}. \tag{6}$$

In [6, prop 3.4] we prove the stronger result:

Proposition 3.3. *The morphism $\mathcal{H}^{2,0}(X) \to F$ is isomorphic to an open subset of a $\mathbb{P}^2$-bundle $\mathbb{P}Q \to F$. In particular, $\mathcal{H}^{2,0}(X)$ is smooth and connected of dimension* 4*. Moreover $\mathbb{P}Q$ is the normalization of $Hilb_{2t+1}(X)$.*

Next we describe the Abel-Jacobi map for conics. By the residuation trick, the Abel-Jacobi map $u_{D_2} : \mathbb{P}Q \to J(X)$ is, up to a fixed translation, the pointwise inverse of the composite $\mathbb{P}Q \xrightarrow{\pi} F \xrightarrow{u} J(X)$. Thus the fibers of $u_{D_2} : \mathbb{P}Q \to J(X)$ are just the fibers of π, i.e. $\mathbb{P}^2$'s.

3.3. **Plane Cubics.** Every curve $C \subset \mathbb{P}^4$ with Hilbert polynomial $3t$ is a plane cubic, and the 2-plane $P = \mathrm{span}(C)$ is unique; we have that $C = X \cap P$. Therefore the Hilbert scheme $\mathrm{Hilb}_{3t}(X)$ is just the Grassmannian $\mathbb{G}(2, \mathbb{P}^4)$ of 2-planes in $\mathbb{P}^4$ and $\mathcal{H}^{3,1}(X)$ is just an open subset of $\mathbb{G}(2, \mathbb{P}^4)$. Since $\mathbb{G}(2, \mathbb{P}^4)$ is rational, it follows that both the total Abel-Jacobi map $\psi : \mathrm{Hilb}_{3t}(X) \times \mathrm{Hilb}_{3t}(X) \rightarrow J(X)$ and the Abel-Jacobi map $i : \mathrm{Hilb}_{3t}(X) \rightarrow J(X)$ are constant maps.

3.4. **Two Useful Results.** Recall that $\psi : F \times F \to J(X)$ is defined as the *difference map* $\psi = u \circ \pi_1 - u \circ \pi_2$. Now we consider the sum map $\psi' = u \circ \pi_1 + u \circ \pi_2$. Alternatively one may define ψ' to be the (Weil extension of the) morphism $(F \times F) - \Delta \rightarrow J(X)$ which is the Abel-Jacobi map corresponding to the flat family $Z \xrightarrow{\pi} (F \times F - \Delta)$ defined as follows. For L_1 and L_2 skew lines, we define $Z_{([L_1],[L_2])}$ to be the disjoint union of L_1 and L_2. If $p \in L_1 \cap L_2$, we define $Z_{([L_1],[L_2])}$ to be the subscheme whose reduced scheme is just the union $L_1 \cup L_2$ and which has an embedded point at p corresponding to the normal direction of $\mathrm{span}(L_1, L_2) \subset T_p X$.

Clearly ψ' commutes with the involution

$$\iota : F \times F \to F \times F, ([L_1], [L_2]) \mapsto ([L_2], [L_1]). \tag{7}$$

So ψ' factors through the quotient $F \times F \rightarrow \mathrm{Sym}^2 F$. We will denote by $\mathrm{Sym}^2 F \xrightarrow{\psi''} J(X)$ the induced morphism. Let us define Θ' to be the scheme-theoretic image of ψ'.

First we prove an easy lemma about cohomology of complex tori. Given a g-dimensional complex torus A and a sequence of nonzero integers $(n_1, \dots, n_r)$, consider the holomorphic map $f = f_{(n_1,\dots,n_r)} : A^r \rightarrow A$ defined by $f(a_1, \dots, a_r) = n_1 a_1 + \dots + n_r a_r$. There is an induced Gysin image map on cohomology $f_* : H^{p+2(r-1)g}(A^r) \rightarrow H^p(A)$. Now by the Künneth formula, the cohomology $H^q(A^r)$ is a direct sum of Künneth components $H^{q_1}(A) \otimes \dots \otimes H^{q_r}(A)$ with $q = q_1 + \dots + q_r$.

Lemma 3.4. *For each integer $r \geq 1$, for each integer $0 \leq p \leq 2g$, and for each decomposition $(q_1, \dots, q_r)$ with $q_1 + \dots + q_r = p + 2(r-1)g$, there is a homomorphism*

$$g = g_{(q_1,\dots,q_r)} : H^{q_1}(A) \otimes \dots \otimes H^{q_r}(A) \rightarrow H^p(A) \tag{8}$$

such that for each sequence of nonzero integers $(n_1, \dots, n_r)$, with f defined as above, the restriction of the Gysin image map

$$f_* : H^{q_1}(A) \otimes \dots \otimes H^{q_r}(A) \rightarrow H^p(A) \tag{9}$$

satisfies $f_* = \left(n_1^{2g-q_1} \cdot \dots \cdot n_r^{2g-q_r}\right) g$.

Proof. This is just a computation. By Poincaré duality, to give the homomorphism g, it is equivalent to give a bilinear pairing

$$(H^{q_1}(A) \otimes \dots \otimes H^{q_r}(A)) \times H^{2g-p}(A) \rightarrow \mathbb{Z}. \tag{10}$$

Moreover, since $H^{2g-p}(A) = \bigwedge^{2g-p} H^1(A)$, it suffices to define the pairing for pure wedge powers $\alpha = \alpha_1 \wedge \dots \wedge \alpha_{2g-p} \in H^{g-p}(A)$. Define S to be the set of functions $\sigma : \{1, 2, \dots, 2g-p\} \rightarrow 1, 2, \dots, r$ such that for each $i = 1, \dots, r$, we have $q_i + \#\sigma^{-1}(i) = 2g$. We define the pairing by taking the wedge product in $H^*(A^r)$ and then taking the degree as follows:

$$\langle \beta = \beta_1 \otimes \dots \otimes \beta_r, \alpha_1 \wedge \dots \wedge \alpha_{2g-p} \rangle = \deg \sum_{\sigma \in S} \pi_1^* \beta_1 \wedge \dots \wedge \pi_r^* \beta_r \wedge \pi_{\sigma(1)}^* \alpha_1 \wedge \dots \wedge \pi_{\sigma(2g-p)}^* \alpha_{2g-p}. \tag{11}$$

The fact that $f_* = \left(n_1^{2g-q_1} \cdot \dots \cdot n_r^{2g-q_r}\right) g$ follows from the projection formula

$$f_*(\beta) \wedge \alpha = f_*(\beta \wedge f^* \alpha) \tag{12}$$

together with the formula

$$f^*(\alpha_1 \wedge \dots \wedge \alpha_r) = f^*(\alpha_1) \wedge \dots \wedge f^*(\alpha_r) = \tag{13}$$

$$\left(\sum_{\sigma(1)=1}^{r} n_{\sigma(1)} \pi_{\sigma(1)}^* (\alpha_1) \right) \wedge \dots \wedge \left(\sum_{\sigma(2g-p)=1}^{r} n_{\sigma(2g-p)} \pi_{\sigma(2g-p)}^* (\alpha_{2g-p}) \right).$$

□

Theorem 3.5. *The image Θ' is a divisor in $J(X)$ which is algebraically equivalent to 3Θ and $\psi' : Sym^2(F) \to \Theta'$ is birational. The singular locus of Θ' has codimension 2. For general $a \in J(X)$, we have that $\Theta' \cap (a+\Theta)$ is irreducible.*

Proof. For divisors on $J(X)$, algebraic equivalence is equivalent to homological equivalence, so we shall establish that Θ' is homologically equivalent to 3Θ.

Using the notation of the previous lemma, both Θ and Θ' are images of the subvariety $u(F) \times u(F) \subset J \times J$ under the morphism $f_{(1,-1)}$ and $f_{(1,1)}$ respectively. By lemma 3.4 we know there is a linear map $g : H^4(J) \otimes H^4(J) \to H^2(J)$ such that

$$f_{(1,-1)*}(P.D.[u(F) \times u(F)]) = (1)^6(-1)^6 g(P.D.[u(F) \times u(F)]), \tag{14}$$
$$f_{(1,1)*}(P.D.[u(F) \times u(F)]) = (1)^6(1)^6 g(P.D.[u(F) \times u(f)])$$

where $P.D.[u(F) \times u(F)]$ is the Poincaré dual of the homology class of $u(F) \times u(F)$. We know the mapping $\psi : F \times F \to \Theta$ has degree 6. So if $\psi' : F \times F \to \Theta'$ has degree d, then it follows that

$$dP.D.[\Theta'] = f_{(1,1)*}(P.D.[u(F) \times u(F)]) = f_{(1,-1)*}(P.D.[u(F) \times u(F)]) = 6P.D.[\Theta]. \tag{15}$$

In other words, the cohomology class of Θ' is equal to $\frac{6}{d}$ times the cohomology class of Θ. Of course d is divisible by 2 because ψ' factors through $F \times F \to \mathrm{Sym}^2(F)$. It remains to prove that d is precisely 2.

Now let $\mathcal{G}' : (\Theta')^{ns} \to \mathbb{P}^{4\vee}$ be the Gauss map defined on the nonsingular locus of Θ'. Of course the differentials $d\psi$ and $d\psi'$ are simply given by $du \circ d\pi_1 - du \circ d\pi_2$ and by $du \circ d\pi_1 + du \circ d\pi_2$ respectively. In particular for each point $(L, M) \in F \times F$, the subspace image$(d\psi) \subset T_0 J$ and image$(d\psi) \subset T_0 J$ are equal, i.e. the composites

$$(F \times F - \Delta) \xrightarrow{\psi} \Theta \xrightarrow{\mathcal{G}} \mathbb{P}^{4\vee} \tag{16}$$
$$F \times F \xrightarrow{\psi'} \Theta' \xrightarrow{\mathcal{G}'} \mathbb{P}^{4\vee} \tag{17}$$

are equal as rational maps; so both are equal to the rational map Φ. Therefore, for each point $([L_1], [L_2])$ whose image under ψ' lies in the nonsingular locus of Θ', we see that the fiber of ψ' containing $([L_1], [L_2])$ is contained in the fiber of Φ containing $([L_1], [L_2])$. So, we are reduced to showing that, for generic $([L_1], [L_2])$, if $\psi'([L_1], [L_2]) = \psi'([L_3], [L_4])$ with L_1, L_2, L_3 and L_4 contained in a smooth hyperplane section of X, then either $([L_1], [L_2]) = ([L_3], [L_4])$ or $([L_2], [L_1]) = ([L_4], [L_3])$.

A bit more generally, suppose that $([L_1], [L_2]) \in U$ is a pair of skew lines. We will show that the fiber of $\psi'^{,-1}([L_1], [L_2]) = \{([L_1], [L_2]), ([L_2], [L_1])\}$. Indeed, suppose that $\psi'([L_1], [L_2]) = \psi'([L_3], [L_4])$ and suppose that $([L_3], [L_4]) \neq ([L_2], [L_1])$. Then $\psi([L_1], [L_4]) = \psi([L_3], [L_2])$. Therefore $\Phi([L_1], [L_4]) = \Phi([L_3], [L_2])$. Let us call this common hyperplane H. Then $L_1 \subset H$ and $L_2 \subset H$. Therefore $H = \Phi([L_1], [L_2])$. Since we have $L_4 \neq L_1$ and $L_3 \neq L_2$, we conclude that $([L_1], [L_4]), ([L_3], [L_2]) \in U$. Therefore, by 3.1 (7), we have either $([L_1], [L_4]) = ([L_3], [L_2])$, or else there exists a line $l \in H$ such that $l \cup L_1 \cup L_2$ is the intersection of X with a $\mathbb{P}^2$. But L_1 and L_2 cannot lie in a common $\mathbb{P}^2$ since they are skew. Therefore we conclude that $([L_1], [L_4]) = ([L_3], [L_2])$, i.e. $([L_3], [L_4]) = ([L_1], [L_2])$.

So we deduce that $d = 2$ and thus Θ' is algebraically equivalent to 3Θ. But we deduce even more. The image of $U - I \cap U$ in $\mathrm{Sym}^2 F$ is smooth, let's call it U'. Since the map $\psi' : U' \to \psi'(U')$ is bijective and since rank$(d\psi')$ = rank$(d\psi) = 4$ on $U - I \cap U$, it follows from Zariski's main theorem [11, p. 288-289] that $\psi'(U')$ is smooth and $U' \xrightarrow{\psi'} \psi' U'$ is an isomorphism. Now the complement of U in $F \times F$ has codimension 2. Therefore $\psi'(F \times F - U)$ has codimension at least 2 inside of Θ'. But in fact $\psi'(I)$ also has codimension 2. Consider the rational map $I \xrightarrow{\rho} F$ defined by sending a pair of incident lines $([L_1], [L_2])$ to the residual line l such that span$(L_1, L_2) \cap X = l \cup L_1 \cup L_2$. By the residuation trick we have that the restriction of ψ' to I equals the composition of ρ with the pointwise negative Abel-Jacobi map $-i$. In particular $\psi'(I)$ is just $-u(F)$ up to translation. So $\psi'(I)$ has codimension 2 inside of Θ'. Therefore $\psi'(U') \subset \Theta'$ has complement of codimension 2. So the singular locus of Θ', has codimension 2.

Since $\Theta' \cap (a + \Theta)$ is a positive dimensional intersection of ample divisors, $\Theta' \cap (a + \Theta)$ is connected. Since $\Theta' \cap (a + \Theta)$ is a complete intersection of Cartier divisors, it follows from Hartshorne's connectedness theorem [2, thereom 18.12] that $\Theta' \cap (a+\Theta)$ is connected in codimension 2. Thus to prove that $\Theta' \cap (a+\Theta)$ is irreducible, it suffices to prove that the singular locus of $\Theta' \cap (a+\Theta)$ has codimension at least 2. By the Bertini-Kleiman theorem [8, theorem III.10.8], we have that for general $a \in J(X)$, the intersection $\Theta' \cap (a+\Theta)$ is smooth away from the intersection of each divisor with the singular locus of the other divisor. But by the last paragraph and by theorem 3.2, we see that the singular loci of Θ and Θ' both have codimension at least 2 (in Θ or Θ' respectively). So it follows that for a general, the singular locus of $\Theta' \cap (a+\Theta)$ has codimension at least 2. Thus we conclude that for general $a \in J(X)$, the intersection $\Theta' \cap (a+\Theta)$ is irreducible.

□

We also use the following enumerative lemma, which is proved in [6, lemma 4.2].

Lemma 3.6. *Suppose that $C \subset X$ is a smooth curve of genus g and degree d. Let $B_C \subset F$ denote the scheme parametrizing lines in X which intersect C in a scheme of degree 2 or more. Define $b(C) = \frac{5d(d-3)}{2} + 6 - 6g$. If B_C is not positive dimensional and if $b(C) \geq 0$, then the degree of B_C is $b(C)$.*

4. Twisted cubics

We now begin in earnest the analysis of the geometry of cubic, quartic and quintic curves on our cubic threefold $X \subset \mathbb{P}^4$. In each case our goal is the proof of the Main Theorem 1.1 for the variety $\mathcal{H}^{d,g}(X)$ parametrizing curves of this degree and genus. We start with the variety $\mathcal{H}^{3,0}(X)$ parametrizing rational curves of degree 3—that is *twisted cubics.*

In [6, theorem 4.4] we prove the following result

Theorem 4.1. *The space $\mathcal{H}^{3,0}(X)$ is a smooth, irreducible 6-dimensional variety.*

We have a morphism

$$\mathcal{H}^{3,0}(\mathbb{P}^4) \xrightarrow{\sigma^{3,0}} \mathbb{P}^{4\vee} \tag{18}$$

defined by sending $[C]$ to $\mathrm{span}(C)$. This morphism makes $\mathcal{H}^{3,0}(\mathbb{P}^4)$ into a locally trivial bundle over $\mathbb{P}^{4\vee}$ with fiber $\mathcal{H}^{3,0}(\mathbb{P}^3)$. Recall from section 3 that we defined $X^\vee \subset \mathbb{P}^{4\vee}$ to be the dual variety of X which parametrizes tangent hyperplanes to X and we defined U to be the complement of $X^\vee$ in $\mathbb{P}^{4\vee}$. Then we define $\mathcal{H}^{3,0}_U(X)$ to be the open subscheme of $\mathcal{H}^{3,0}(X)$ which parametrizes twisted cubics, C, in X such that $\sigma^{3,0}([C]) \in U$. By the graph construction we may consider $\mathcal{H}^{3,0}_U(X)$ as a locally closed subvariety of $\mathrm{Hilb}_{3t+1}(X) \times U$. Let $\overline{\mathcal{H}} \subset \mathrm{Hilb}_{3t+1}(X) \times U$ denote the closure of $\mathcal{H}^{3,0}_U(X)$ with the reduced induced scheme structure. Denote by $\overline{\mathcal{H}} \xrightarrow{f} U(X)$ the projection map.

Theorem 4.2. *Let $\overline{\mathcal{H}} \xrightarrow{f''} U' \xrightarrow{f'} U(X)$ be the Stein factorization of $\overline{\mathcal{H}} \xrightarrow{f} U(X)$. Then $\overline{\mathcal{H}} \xrightarrow{f''} U'$ is isomorphic to a $\mathbb{P}^2$-bundle $\mathbb{P}_{U'}(E) \xrightarrow{\pi} U'$ with E a locally free sheaf of rank 3. And $U' \xrightarrow{f'} U$ is an unramified finite morphism of degree 72. Moreover, the Abel-Jacobi map $\overline{\mathcal{H}} \xrightarrow{i} J(X)$ factors as $\overline{\mathcal{H}} \xrightarrow{f''} U' \xrightarrow{i'} J(X)$ where $U' \xrightarrow{i'} J(X)$ is a birational morphism of U' to a translate of Θ.*

Proof. This theorem is [6, theorem 4.5], and is proved there. For brevity's sake, we recall just the sketch of the proof.

One can form the subvariety $\mathcal{X} \subset U \times X$ as the universal smooth hyperplane section of X. Each fiber of $\mathcal{X} \to U$ is a smooth cubic surface. We associate to this family of cubic surfaces the finite étale morphism

$$\rho : \mathrm{Pic}^{3,0}(\mathcal{X}/U) \to U \tag{19}$$

whose fiber over a point $[H] \in U$ is simply the set of divisor classes D on the smooth cubic surface $H \cap U$ such that $D.D = 1$ and $D.h = 3$ with h the hyperplane class. Now the Weyl group $\mathcal{W}(E_6)$ acts transitively on the set of such divisor classes D. It follows that with respect to the model of a cubic surface as $\mathbb{P}^2$ blown up at 6 points, each divisor class D above corresponds to the class of a line on $\mathbb{P}^2$. From this it follows that the induced morphism

$$g : \mathcal{H}^{3,0}(X) \to \mathrm{Pic}^{3,0}(\mathcal{X}/U)) \tag{20}$$

is surjective and isomorphic to an open subset of a $\mathbb{P}^2$-bundle on $\mathrm{Pic}^{3,0}(\mathcal{X}/U)$. In particular, since $u : \mathcal{H}^{3,0}(X) \to J(X)$ contracts all rational curves, there is an induced morphism $u' : \mathrm{Pic}^{3,0}(\mathcal{X}/U) \to J(X)$ such that $u = u' \circ g$.

It only remains to show that u' maps $\mathrm{Pic}^{3,0}(\mathcal{X}/U)$ birationally to its image. This is proved by analyzing "Z"s of lines, i.e. configurations of lines on a cubic surface $H \cap X$ whose dual graph is just the connected graph with 3 vertices and no loops. Using 3.1, part 5, we conclude that two "Z"s have the same image in $J(X)$ iff they are in the same linear equivalence class on $H \cap X$. ☐

Corollary 4.3. *The Abel-Jacobi map $u_{3,0} : \mathcal{H}^{3,0}(X) \to J(X)$ dominates a translate of Θ and is birational to a $\mathbb{P}^2$-bundle over its image.*

Proof. Since $\mathcal{H}^{3,0}(X)$ is irreducible, $\mathcal{H}^{3,0}_U(X)$ is dense in $\mathcal{H}^{3,0}(X)$. ☐

4.1. **Quartic Elliptic Curves.** Recall that the normalization of $\mathrm{Hilb}_{2t+1}(X)$ is isomorphic to the $\mathbb{P}^2$-bundle $\mathbb{P}Q \to F$ which parametrizes pairs (L, P) which $L \subset X$ a line and $P \subset \mathbb{P}^4$ a 2-plane containing L. Let $A \xrightarrow{g} \mathbb{P}Q$ denote the $\mathbb{P}^1$-bundle which parametrizes triples (L, P, H) with H a hyperplane containing P. Let $I_{4,1} \xrightarrow{h} A$ denote the $\mathbb{P}^4$-bundle parametrizing 4-tuples (L, P, H, Q) where $Q \subset H$ is a quadric surface containing the conic $C \subset X \cap P$. Notice that $I_{4,1}$ is smooth and connected of dimension $4 + 1 + 4 = 9$.

Let $D \subset I_{4,1} \times X$ denote the intersection of the universal quadric surface over $I_{4,1}$ with $I_{4,1} \times X \subset I_{4,1} \times \mathbb{P}^4$. Then D is a local complete intersection scheme. By the Lefschetz hyperplane theorem, X contains no quadric surfaces; therefore $D \to I_{4,1}$ has constant fiber dimension 1 and so is flat. Let $D_1 \subset I_{4,1} \times X$ denote the pullback from $\mathbb{P}Q \times X = \mathrm{Hilb}_{2t+1}(X) \times X$ of the universal family of conics. Since $I_{4,1} \times X \to \mathbb{P}Q \times X$ is smooth and the universal family of conics is a local complete intersection which is flat over $\mathbb{P}Q$, we conclude that also D_1 is a local complete intersection which is flat over $I_{4,1}$. Clearly $D_1 \subset D$. Thus by corollary [6, corollary 2.7], we see that the residual D_2 of $D_1 \subset D$ is Cohen-Macaulay and flat over $I_{4,1}$.

By the base-change property in corollary [6, corollary 2.7], we see that the fiber of $D_1 \to I_{4,1}$ over a point (L, P, H, Q) is simply the residual of $C \subset Q \cap X$. If we choose Q to be a smooth quadric, i.e. $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$, then $C \subset Q$ is a divisor of type $(1, 1)$ and $X \cap Q \subset Q$ is a divisor of type $(3, 3)$. Thus the residual curve E is a divisor of type $(2, 2)$, i.e. a quartic curve of arithmetic genus 1. Thus $D_2 \subset I_{4,1} \times X$ is a family of connected, closed subschemes of X with Hilbert polynomial $4t$. So we have an induced map $f : I_{4,1} \to \mathrm{Hilb}_{4t}(X)$.

Proposition 4.4. *The image of the morphism above $f : I_{4,1} \to Hilb_{4t}(X)$ is the closure $\overline{\mathcal{H}^{4,1}}(X)$ of $\mathcal{H}^{4,1}(X)$. Moreover the open set $f^{-1}\mathcal{H}^{4,1}(X) \subset I_{4,1}$ is a $\mathbb{P}^1$-bundle over $\mathcal{H}^{4,1}(X)$. Thus $\mathcal{H}^{4,1}(X)$ is smooth and connected of dimension 8.*

Proof. This is [6, proposition 5.1]. ☐

If (L, P, H, Q) is a point in the fiber over $[E]$, then $H = \mathrm{span}(E)$. And since there is no rational curve in F, we also have that L is constant in the fiber. So we have a well-defined morphism $m : \mathcal{H}^{4,1}(X) \to \mathbb{P}Q^\vee$, where $\mathbb{P}Q^\vee$ is the $\mathbb{P}^2$-bundle over F parametrizing pairs $([L], [H]), L \subset H$. For a general H, the intersection $Y = H \cap X$ is a smooth cubic surface. And the fiber $m^{-1}([L], [H])$ is an open subset of the complete linear series $|\mathcal{O}_Y L + h|$, where h is the hyperplane class on Y. Thus $m : \mathcal{H}^{4,1}(X) \to \mathbb{P}Q^\vee$ is a morphism of smooth connected varieties which is birational to a $\mathbb{P}^4$-bundle. Composing m with the projection $\mathbb{P}Q^\vee$ yields a morphism $n : \mathcal{H}^{4,1}(X) \to F$ which is birational to a $\mathbb{P}^4$-bundle over a $\mathbb{P}^2$-bundle.

Corollary 4.5. *By the residuation trick, we conclude the Abel-Jacobi map $u_{4,1} : \mathcal{H}^{4,1}(X) \to J(X)$ is equal, up to a fixed translation, to the composite:*

$$\mathcal{H}^{4,1}(X) \xrightarrow{\quad n \quad} F \xrightarrow{\quad u_{1,0} \quad} J(X). \tag{21}$$

Thus the general fiber of $u_{4,1}$ equals the general fiber of n, and so is isomorphic to an open subset of a $\mathbb{P}^4$-bundle over $\mathbb{P}^2$.

5. Cubic scrolls and applications

5.1. **Preliminaries on cubic scrolls.**
In the next few sections we will use residuation in a cubic surface scroll. We start by collecting some basic facts about these surfaces.

There are several equivalent descriptions of cubic scrolls.

(1) A cubic scroll $\Sigma \subset \mathbb{P}^4$ is a connected, smooth surface with Hilbert polynomial $P(t) = \frac{3}{2}t^2 + \frac{5}{2}t + 1$.
(2) A cubic scroll $\Sigma \subset \mathbb{P}^4$ is the determinantal variety defined by the 2×2 minors of a matrix of linear forms:
$$\begin{bmatrix} L_1 & L_2 & L_3 \\ M_1 & M_2 & M_3 \end{bmatrix} \tag{22}$$
such that for each row or column, the linear forms in that row or column are linearly independent
(3) A cubic scroll $\Sigma \subset \mathbb{P}^4$ is the *join* of an isomorphism $\phi : L \to C$. Here $L \subset \mathbb{P}^4$ is a line and $C \subset \mathbb{P}^4$ such that $L \cap \text{span}(C) = \emptyset$. The join of ϕ is defined as the union over all $p \in L$ of the line $\text{span}(p, \phi(p))$.
(4) A cubic scroll $\Sigma \subset \mathbb{P}^4$ is the image of a morphism $f : \mathbb{P}E \to \mathbb{P}^4$ where E is the rank 2 vector bundle on $\mathbb{P}^1$, $E = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$, and the morphism $f : \mathbb{P}E \to \mathbb{P}^4$ is such that $f^*\mathcal{O}_{\mathbb{P}^4}(1) = \mathcal{O}_{\mathbb{P}E}(1)$ and the pullback map $H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(1)) \to H^0(\mathbb{P}E, \mathcal{O}_{\mathbb{P}E}(1)$ is an isomorphism.
(5) A cubic scroll $\Sigma \subset \mathbb{P}^4$ is as a minimal variety, i.e. $\Sigma \subset \mathbb{P}^4$ is any smooth connected surface with $\text{span}(\Sigma) = \mathbb{P}^4$ which has the minimal possible degree for such a surface, namely $\deg(\Sigma) = 3$.
(6) A cubic scroll $\Sigma \subset \mathbb{P}^4$ is a smooth surface residual to a 2-plane Π in the base locus of a pencil of quadric hypersurfaces which contain Π.

From the fourth description $\Sigma = f(\mathbb{P}E)$ we see that $\text{Pic}(\Sigma) = \text{Pic}(\mathbb{P}E) \cong \mathbb{Z}^2$. Let $\pi : \mathbb{P}E \to \mathbb{P}^1$ denote the projection morphism and let $\sigma : \mathbb{P}^1 \to \mathbb{P}E$ denote the unique section whose image $D = \sigma(\mathbb{P}^1)$ has self-intersection $D.D = -1$. Then $f(D)$ is a line on Σ called the *directrix*. And for each $t \in \mathbb{P}^1$, $f(\pi^{-1}(t))$ is a line called a *line of the ruling* of Σ. Denote by F the divisor class of any $\pi^{-1}(t)$. Then $\text{Pic}(\Sigma) = \mathbb{Z}\{D, F\}$ and the intersection pairing on Σ is determined by $D.D = -1, D.F = 1, F.F = 0$. The hyperplane class is $H = D + 2F$ and the canonical class is $K = -2D - 3F$.

Using the fourth description of a cubic scroll, we see that any two cubic scrolls differ only by the choice of the isomorphism $H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(1)) \to H^0(\mathbb{P}E, \mathcal{O}_{\mathbb{P}E}(1))$. Therefore any two cubic scrolls are conjugate under the action of $\text{PGL}(5)$. So the open set $U \subset \text{Hilb}_{P(t)}(\mathbb{P}^4)$ parametrizing cubic scrolls is a homogeneous space for $\text{PGL}(5)$, in particular it is smooth, connected and rational. So the Abel-Jacobi map $U \to J(X)$ associated to the family of intersections $\Sigma \cap X \subset X$ is a constant map.

5.2. **Cubic Scrolls and Quartic Rational Curves.**
Recall that $\text{Pic}(\Sigma) = \mathbb{Z}\{D, F\}$ where D is the directrix and F is the class of a line of ruling. The intersection product is given by $D^2 = -1, D.F = 1, F^2 = 0$. The canonical class is given by $K_\Sigma = -2D - 3F$ and the hyperplane class is given by $H = D + 2F$. The linear system $|F|$ is nef because it is the pullback of $\mathcal{O}_{\mathbb{P}^1}(1)$ under the projection $\pi : \Sigma \to \mathbb{P}^1$. Similarly, $|D + F|$ is nef because it contains all the conics obtained as the residuals to lines of the ruling in $|H|$. Thus for any effective curve class $aD + bF$ we have the two inequalities $a = (aD + bF).F \geq 0$, $b = (aD + bF).(D + F) \geq 0$.

Suppose that $C \subset \Sigma$ is an effective divisor of degree 4 and arithmetic genus 0. By the adjunction formula
$$K_\Sigma.[C] + [C].[C] = 2p_a - 2 = -2. \tag{23}$$
So if $[C] = aD + bF$, then we have the conditions
$$a \geq 0, b \geq 0, a + b = 4, a^2 - 2ab + a + 2b = 2. \tag{24}$$
It is easy to check that there are precisely two solutions $[C] = 2D + 2F, [C] = D + 3F$. We will see that both possibilities occur and describe some constructions related to each possibility.

Lemma 5.1. *Let $C \subset \mathbb{P}^4$ be a smooth quartic rational curve and let $V \subset |\mathcal{O}_C(2)|$ be a pencil of degree 2-divisors on C without basepoints. There exists a unique map of a Hirzebruch surface $\mathbf{F}_1$, $f : \Sigma \to \mathbb{P}^4$ such that $(f^*H)^2 = 3$ and a factorization $i : C \to \Sigma$ of $C \to \mathbb{P}^4$ such that $i(C) \sim 2D + 2F$ and the pencil of degree 2 divisors $F \cap C$ is the pencil V.*

Proof. This is [6, lemma 6.7]. □

Remark While we are at it, let's mention a specialization of the construction above, namely what happens if V is not basepoint free. Then $V = p + |\mathcal{O}_C(1)|$ where $p \in C$ is some basepoint. Consider the projection morphism $f : \mathbb{P}^4 \to \mathbb{P}^3$ obtained by projection from p (this is a rational map undefined at p). The image of C is a rational cubic curve B (possibly a singular plane cubic). Consider the cone Σ' in $\mathbb{P}^4$ over B with vertex p. This surface contains C. If we blowup $\mathbb{P}^4$ at p, then the proper transform of Σ' in $\widetilde{\mathbb{P}^4}$ is a surface whose normalization Σ is a Hirzebruch surface $\mathbf{F}_3$ (normalization is only necessary if B is a plane curve). The directrix D of Σ is the pullback of the exceptional divisor of $\widetilde{\mathbb{P}^4}$. The inclusion $C \subset \Sigma'$ induces a factorization $i : C \to \Sigma$ of $C \to \mathbb{P}^4$, and $[i(C)] = D + 4F$. The intersection of D and $i(C)$ is precisely the point p. And the linear system $i^*|F|$ is exactly $|\mathcal{O}_C(1)|$.

Next we consider the case of a rational curve $C \subset \Sigma$ such that $[C] = D + 3F$.

Lemma 5.2. *Let $C \subset \mathbb{P}^4$ be a smooth quartic rational curve and let $L \subset \mathbb{P}^4$ be a line such that $L \cap C = Z$ is a degree 2 divisor. Let $\phi : C \to L$ be an isomorphism such that $\phi(Z) = Z$ and $\phi|_Z$ is the identity map. Then there exists a unique triple (h, i, j) where $h : \Sigma \to \mathbb{P}^4$ is a finite map of a Hirzebruch surface $\Sigma \equiv \mathbf{F}_1$, and $i : C \to \mathbb{P}^4$, $j : L \to \mathbb{P}^4$ are factorizations of $C \to \mathbb{P}^4, L \to \mathbb{P}^4$ such that $j(L) = D$ is the directrix, such that $[i(C)] = D + 3F$ and such that the composition of $i : C \to \Sigma$ with the projection $\pi : \Sigma \to D$ equals $j \circ \phi$.*

Proof. This is [6, lemma 6.8]. ☐

5.3. **Cubic Scrolls and Quintic Elliptics.**

Recall that our fourth description of a cubic scroll was the image of a morphism $f : \Sigma \to \mathbb{P}^4$ where Σ is the Hirzebruch surface $\mathbb{F}_1$, $f^*\mathcal{O}(1) \sim \mathcal{O}_\Sigma(1) = \mathcal{O}_{\mathbb{P}E}(D + 2F)$, and $f : \Sigma \to \mathbb{P}^4$ is given by the complete linear series of $\mathcal{O}_\Sigma(D + 2F)$. In the next sections it will be useful to weaken this last condition.

Definition 5.3. A *cubic scroll* in $\mathbb{P}^n$ is a finite morphism $f : \Sigma \to \mathbb{P}^n$ where Σ is isomorphic to the Hirzebruch surface $\mathbb{F}_1$ and such that $f^*\mathcal{O}_{\mathbb{P}^n}(1)$ is isomorphic to $\mathcal{O}_\Sigma(D + 2F)$.

Let $H = D + 2F$ denote the pullback of the hyperplane class. Now suppose that $E \subset \Sigma$ is an effective Cartier divisor with $p_a(E) = 1$ and $E.H = 5$. Since F and $D + F$ are effective and move, we have $E.F, E.(D + F) \geq 0$. Writing $E = aD + bF$ we see (a, b) satisfies the relations $a, b \geq 0, a + b = 5$ and $a(b - 3) + b(a - 2) - a(a - 2) = 0$. These relations give the unique solution $E = 2D + 3F = -K$. In particular, if E is smooth then $\pi : E \to \mathbb{P}^1$ is a finite morphism of degree 2, i.e. a g^1_2 on E. Thus a pair $(f : \Sigma \to \mathbb{P}^n, E \subset \Sigma)$ of a cubic scroll and a quintic elliptic determines a pair $(g : E \to \mathbb{P}^n, \pi : E \to \mathbb{P}^1)$ where $g : E \to \mathbb{P}^n$ is a quintic elliptic and $\pi : E \to \mathbb{P}^1$ is a degree 2 morphism.

Suppose we start with a pair $(g : E \to \mathbb{P}^n, \pi : E \to \mathbb{P}^1)$ where $g : E \to \mathbb{P}^n$ is an embedding of a quintic elliptic curve and $\pi : E \to \mathbb{P}^1$ is a degree 2 morphism. Consider the rank 2 vector bundle $\pi_* g^* \mathcal{O}_{\mathbb{P}^n}(1)$.

Lemma 5.4. *Suppose E is an elliptic curve and $\pi : E \to \mathbb{P}^1$ is a degree 2 morphism. Suppose L is an invertible sheaf on E of degree d. Then we have*

$$\pi_* L \cong \begin{cases} \mathcal{O}_{\mathbb{P}^1}(e) \oplus \mathcal{O}_{\mathbb{P}^1}(e-1) & d = 2e+1, \\ \mathcal{O}_{\mathbb{P}^1}(e) \oplus \mathcal{O}_{\mathbb{P}^1}(e-2) & d = 2e, L \cong \pi^*\mathcal{O}_{\mathbb{P}^1}(e), \\ \mathcal{O}_{\mathbb{P}^1}(e-1) \oplus \mathcal{O}_{\mathbb{P}^1}(e-1) & d = 2e, L \ncong \pi^*\mathcal{O}_{\mathbb{P}^1}(e) \end{cases} \tag{25}$$

Proof. This is [6, lemma 6.10]. ☐

By the lemma we see that the vector bundle $G := \pi_* g^* \mathcal{O}_{\mathbb{P}^n}(1)$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(2)$. Associated to the linear series $\mathcal{O}_E^{n+1} \to g^*\mathcal{O}_{\mathbb{P}^n}(1)$ defining the embedding g, we have the push-forward linear series $\mathcal{O}_{\mathbb{P}^1}^{n+1} \to G$. Since g is an embedding, for each pair of points $\{p, q\} \subset E$ (possibly infinitely near), we have that $\mathcal{O}_E^{n+1} \to g^*\mathcal{O}_{\mathbb{P}^n}(1)|_{\{p,q\}}$ is surjective. In particular taking $\{p, q\} = \pi^{-1}(t)$ for $t \in \mathbb{P}^1$, we conclude that $\mathcal{O}_{\mathbb{P}^1}^{n+1} \to F|_t$ is surjective. Thus we have an induced morphism $\mathbb{P}G^\vee \to \mathbb{P}^n$ which pulls back $\mathcal{O}_{\mathbb{P}^n}(1)$ to $\mathcal{O}_{\mathbb{P}G^\vee}(1)$. Let us denote $\Sigma := \mathbb{P}G^\vee$ and let us denote the morphism by $f : \Sigma \to \mathbb{P}^n$. Abstractly Σ is isomorphic to $\mathbb{F}_1$ and $f : \Sigma \to \mathbb{P}^n$ is a cubic scroll.

The tautological map $\pi^*\pi_* g^*\mathcal{O}_{\mathbb{P}^n}(1) \to g^*\mathcal{O}_{\mathbb{P}^n}(1)$ is clearly surjective. Thus there is an induced morphism $h : E \to \Sigma$. Chasing definitions, we see that $g = f \circ h$. So we conclude that given a pair $(g : E \to \mathbb{P}^n, \pi : E \to \mathbb{P}^1)$ as above, we obtain a pair $(f : \Sigma \to \mathbb{P}^n, h : E \to \Sigma)$. Thus we have prove the following:

Lemma 5.5. *There is an equivalence between the collection of pairs* $(f : \Sigma \rightarrow \mathbb{P}^n, h : E \rightarrow \Sigma)$ *with* $f : \Sigma \rightarrow \mathbb{P}^n$ *a cubic scroll and* $f \circ h : E \rightarrow \mathbb{P}^n$ *an embedded quintic elliptic curve and the collection of pairs* $(g : E \rightarrow \mathbb{P}^n, \pi : E \rightarrow \mathbb{P}^1)$ *where* $g : E \rightarrow \mathbb{P}^n$ *is an embedded quintic elliptic curve and* $\pi : E \rightarrow \mathbb{P}^1$ *is a degree 2 morphism.*

Stated more precisely, this gives an isomorphism of the parameter schemes of such pairs, but we don't need such a precise result.

5.4. **Cubic Scrolls and Quintic Rational Curves.**

If one carries out the analogous computations as at the beginning of subsection 5.2 one sees that the only effective divisor classes $aD + bF$ on a cubic scroll Σ with degree 5 and arithmetic genus 0 are $D + 4F$ and $3D+2F$. But the divisor class $3D+2F$ cannot be the divisor of an irreducible curve because $(3D+2F).D = -1$. Thus if $C \subset \Sigma$ is an irreducible curve of degree 5 and arithmetic genus 0, then $[C] = D + 4F$.

Lemma 5.6. *Let* $C \subset \mathbb{P}^4$ *be a smooth quintic rational curve and let* $L \subset \mathbb{P}^4$ *be a line such that* $L \cap C$ *is a degree 3 divisor* Z. *Let* $\phi : C \to L$ *be an isomorphism such that* $\phi(Z) = Z$ *and* $\phi|_Z$ *is the identity map. Then there exists a unique triple* (h, i, j) *such that* $h : \Sigma \rightarrow \mathbb{P}^4$ *is a finite map of a Hirzebruch surface* $\Sigma \equiv \mathbb{P}^4$, *and* $i : C \rightarrow \mathbb{P}^4, j : L \rightarrow \mathbb{P}^4$ *are factorizations of* $C \rightarrow \mathbb{P}^4, L \rightarrow \mathbb{P}^4$ *such that* $j(L) = D$ *is the directrix, such that* $[i(C)] = D + 4F$ *and such that the composition of* i *with the projection* $\pi : \Sigma \to D$ *equals* $j \circ \phi$.

Proof. This is [6, lemma 6.12]. □

6. Quartic Rational Curves

In this section we will prove that the Abel-Jacobi map $u_{4,0} : \mathcal{H}^{4,0}(X) \to J(X)$ is dominant and the general fiber is irreducible of dimension 3. In a later section we will prove that the general fiber is unirational.

The construction we use to understand quartic rational curves is as follows. For any quartic rational curve $C \subset X$, define $A_C \subset F$ to be the scheme parametrizing 2-secant lines to C. By lemma 3.6, A_C is either positive-dimensional or else has length 16. Define $I \subset \mathcal{H}^{4,0}(X) \times \mathrm{Hilb}_2(F)$ to be the space of pairs $([C], [Z])$ where $Z \subset A_C$ is a 0-dimensional length 2 subscheme of A_C. Denote by $\Lambda \subset Z \times \mathbb{P}^4$ the flat family of lines determined by $Z \subset F$.

Definition 6.1. We say Z is *planar* if there exists a 2-plane $P \subset \mathbb{P}^4$ such that $\Lambda \subset Z \times P$. We say Z is *nonplanar* if Z is not planar.

By lemma 7.1, lemma 7.2 and theorem 7.3 of [6], we have the following result

Theorem 6.2. *The morphism* $I \to \mathcal{H}^{4,0}(X)$ *is generically finite,* I *is irreducible of dimension* 8 *and therefore* $\mathcal{H}^{4,0}(X)$ *is irreducible of dimension* 8. *For a general pair* $([C], [Z]) \in I$, C *is nondegenerate and* Z *is reduced and nonplanar.*

Let $([C], [Z]) \in I$ be a general pair. Now we may also consider Z as a subscheme of $\mathrm{Sym}^2(C) \cong \mathbb{P}^2$. Since Z is non-planar, in particular the span of $Z \subset \mathbb{P}^2$ yields a pencil of degree 2 divisors on C without basepoints. By lemma 5.1, there is a cubic scroll $f : \Sigma \rightarrow \mathbb{P}^4$ along with a factorization $i : C \rightarrow \mathbb{P}^4$ such that the pencil of degree 2 divisors on C is just the pencil of intersections of C with the lines of ruling of Σ. The residual of $C \cup \Lambda$ in $\Sigma \cap X$ is a curve in Σ of degree 3 and arithmetic genus 0. As a corollary of the proof of [6, theorem 7.3], for $([C], [Z])$ a general pair, the residual curve D is a twisted cubic curve. As in theorem 4.2, let $U' \subset J(X)$ denote the Abel-Jacobi image of the locus of twisted cubics $D \subset X$ such that $\mathrm{span}(D) \cap X$ is a smooth cubic surface. And recall from theorem 3.5 that the Abel-Jacobi map $\mathrm{Hilb}_2(F) \to \Theta'$ is birational. Define $R = \Theta' \times U'$ and define a rational transformation

$$h : I \to R, ([C], [Z]) \mapsto (u[Z], u([D])). \tag{26}$$

Lemma 6.3. *The rational transformation* $h : I \to R$ *is birational.*

Proof. By Zariski's main theorem, it suffices to prove that for the general element $(x, y) \in R$, there is a unique pair $([C], [Z])$ such that $h([C], [Z]) = (x, y)$. Now $x = \{L_1, L_2\}$ is a general pair of disjoint lines in X. And y is a linear equivalence class $|D|$ of twisted cubics on a general hyperplane section $H \cap X$. Then $L_i \cap H = \{p_i\}$ is a general point on $H \cap X$ (since every point of X lies on a line, we may assume this point is general on $H \cap X$). And there is a unique twisted cubic $D \subset H \cap X$ in the linear equivalence class $|D|$ and which contains the two points p_1 and p_2. If $h([C], [Z]) = (x, y)$, then the residual to $C \cup L_1 \cup L_2$ in Σ can only be D.

Now consider the hyperplane $H' = \mathrm{span}(L_1, L_2)$. The intersection $H' \cap D$ consists of three points p_1, p_2 and a third point q. Moreover the directrix M of Σ is contained in H. And by a divisor class calculation, $M \cap D$ consists of a point other than p_1, p_2. The only possibility is that $M \cap D = \{q\}$. Notice that given two skew lines L_1, L_2 in a 3-plane H' and given a point $q \in H'$ not lying on $L_1 \cup L_2$, there is a unique line $M \subset H'$ which contains q and intersects each of L_1, L_2. Indeed, if we project from q then L_1, L_2 project to distinct lines in $H/q \cong \mathbb{P}^2$ which intersect in a unique point. And M is simply the cone over this point. So we conclude that the directrix line M is uniquely determined by (x, y).

Finally, projection $\Sigma \to M$ to the directrix determines an isomorphism $\phi : D \to M$ such that for each $r \in D, \mathrm{span}(r, \phi(r))$ is a line of the ruling of Σ, in particular $\phi(q) = q$. Conversely, given a line M which intersects C in one point q and given an isomorphism $\phi : C \to M$ such that $\phi(q) = q$, then the union of the lines $\mathrm{span}(r, \phi(r))$ is a cubic scroll Σ. Thus to uniquely specify the cubic scroll Σ, we have only to determine ϕ. But notice also that we already know $\phi(p_1), \phi(p_2)$ are the unique points of intersection of M with L_1 and L_2 respectively. Thus ϕ is also uniquely determined by (x, y). Altogether we conclude that Σ is uniquely determined by (x, y). But then we can recover/construct C as the residual to $D \cup L_1 \cup L_2$ in $\Sigma \cap X$. This proves that h is birational. □

Theorem 6.4. *The composite of* $I \to \mathcal{H}^{4,0}(X)$ *with the Abel-Jacobi map* $u_{4,0} : \mathcal{H}^{4,0}(X) \to J(X)$ *is dominant and the general fiber is irreducible. Therefore* $u_{4,0}$ *is dominant and the general fiber is irreducible.*

Proof. By the residuation trick, the composite equals (as a rational map) the pointwise inverse of the composition

$$I \xrightarrow{h} R = \Theta' \times U' \to J(X) \tag{27}$$

where the second map is the restriction to $\Theta' \times U' \subset J(X) \times J(X)$ of the addition map. Clearly $\Theta' + U' = J(X)$ since Θ' is a divisor. $U' \subset \Theta$ is a Zariski-dense open set, and Θ is not contained in any translate of Θ'. Thus $I \to J(X)$ is dominant.

Using the fact that Θ is a symmetric divisor, we see that the general fiber of the map $\Theta' \times \Theta \to J(X)$ is an intersection $\Theta \cap (a + \Theta')$. By theorem 3.5, for general a this intersection is irreducible. Thus we conclude that the general fiber of $I \to J(X)$ is irreducible. This proves the theorem. □

7. Quintic Elliptics

In this section we will prove that the Abel-Jacobi map $u_{5,1} : \mathcal{H}^{5,1}(X) \to J(X)$ is dominant and the general fiber is an irreducible 5-fold. In the next section we will see that the fibers are unirational.

The construction we use to understand quintic elliptics is as follows. Define $g : \widetilde{H} \to \mathcal{H}^{5,1}(X)$ to be the relative Pic^2 of the universal family of elliptic curves $\mathcal{C} \to \mathcal{H}^{5,1}(X)$. By lemma 5.5 $\widetilde{H}$ is also the parameter space for pairs (f, h) where $f : \Sigma \to \mathbb{P}^4$ is a generalized cubic scroll and $h : C \to \Sigma$ is a curve such that $f(h(C)) \subset X$ is a smooth quintic elliptic. The residual to C in $\Sigma \cap X$ is a curve C' of degree 4 and arithemtic genus 0.

Theorem 7.1. *The scheme* $\widetilde{H}$ *is irreducible of dimension* 11*. For a general pair* $(f, h) \in \widetilde{H}$*, the residual curve* D *is a smooth, nondegenerate, quartic rational curve.*

Proof. This follows from the proof of [6, theorem 8.1]. □

We have an induced rational transformation $g' : \widetilde{H} \to \mathcal{H}^{4,0}(X)$ which sends a pair (f, h) to the residual curve C'. Also by lemma 5.2, we see that $g' : \widetilde{H} \to \mathcal{H}^{4,0}(X)$ is (birationally) the parameter space for pairs (f', h') where $f' : \Sigma \to \mathbb{P}^4$ is a cubic scroll and $h' : C' \to \Sigma$ is a curve which intersects the directrix in a degree 2 divisor and lines of the ruling in a degree 1 divisor, and such that $f'(h'(C')) \subset X$ is a quartic rational curve. Given such a pair, the residual to C' in $\Sigma \cap X$ is the quintic elliptic C we started with, so both descriptions of $\widetilde{H}$ are equivalent.

Notice that by lemma 5.2, the fiber of g' over a general point $[C']$ is the irreducible (rational) variety parametrizing pairs (D, ϕ) where D is a 2-secant line to C' and where $\phi : C' \to D$ is an isomorphism such that ϕ is the identity on $C' \cap D$.

Theorem 7.2. *The composite of $g : \widetilde{H} \to \mathcal{H}^{5,1}(X)$ and the Abel-Jacobi map $u_{5,1} : \mathcal{H}^{5,1}(X) \to J(X)$ is dominant and the general fiber is an irreducible 6-fold. Therefore the Abel-Jacobi map $u_{5,1} : \mathcal{H}^{5,1}(X) \to J(X)$ is dominant and the general fiber is an irreducible 5-fold.*

Proof. The space of all embedded cubic scrolls $f : \Sigma \to \mathbb{P}^4$ is clearly unirational, in fact it is a homogeneous space for PGL(5) and so it is even rational. So the Abel-Jacobi map is constant on the family of complete intersections f^*X. By the residuation trick, the following two birational transformations are pointwise (additive) inverses:

$$\widetilde{H} \xrightarrow{g} \mathcal{H}^{5,1}(X) \xrightarrow{u_{5,1}} J(X) \tag{28}$$
$$\widetilde{H} \xrightarrow{g'} \mathcal{H}^{4,0}(X) \xrightarrow{u_{4,0}} J(X).$$

By theorem 6.4, $u_{4,0}$ is dominant and the general fiber is irreducible. We have seen that $\widetilde{H} \to \mathcal{H}^{4,0}(X)$ is dominant and the general fiber is irreducible. Thus one, and hence both of the morphisms $\widetilde{H}_1 \to J(X)$ are dominant and the general fiber is irreducible. Since $\widetilde{H} \to \mathcal{H}^{5,1}(X)$ is dominant, we conclude that $u_{5,1}$ is dominant and the general fiber is irreducible. ☐

8. Double Residuation and Unirationality of the Fibers

In the last section we introduced the space $\widetilde{H}$ which parametrizes pairs $(f : \Sigma \to \mathbb{P}^4, h : E \to \Sigma)$ where $f : \Sigma \to \mathbb{P}^4$ is a cubic scroll and $E \subset \Sigma$ is an elliptic curve such that $f \circ h : E \to \mathbb{P}^4$ is an embedding of E as a quintic elliptic curve in X. Equivalently $\widetilde{H}$ parametrizes pairs $(f : \Sigma \to \mathbb{P}^4, k : C \to \Sigma)$ where C is the rational curve residual to E in f^*X and $f \circ k : C \to X$ is a quartic rational curve. In this section we will use $\widetilde{H}$ to prove that the fibers of $u_{4,0}$ and $u_{5,1}$ are unirational.

We need a partial compactification of $\widetilde{H}$. Let $\overline{H}^{5,1} \subset \mathrm{Hilb}_{5t}(X)$ denote the open subset of the closure of $\mathcal{H}^{5,1}(X)$ which parametrizes Cohen-Macaulay curves (i.e. curves with no embedded points). Let $P(t)$ be the numerical polynomial $P(t) = \frac{3}{2}t^2 + \frac{5}{2}t + 1$. Let $M \subset \mathrm{Hilb}_{P(t)}(\mathbb{P}^4)$ denote the open subscheme parametrizing connected, reduced, local complete intersection subschemes of $\mathbb{P}^4$ with Hilbert polynomial $P(t)$. The space M contains an open subset parametrizing embedded cubic scrolls, but M also parametrizes mild degenerations of embedded cubic scrolls. We will refer to the schemes parametrized by M as *generalized cubic scrolls.* Let $N \subset M \times \overline{H}^{5,1}$ denote the locally closed subscheme parametrizing pairs (Σ, E) such that $X \cap \Sigma$ is a reduced Weil divisor and $E \subset X \cap \Sigma$. Notice that the Lefschetz hyperplane theorem shows that X contains no generalized cubic scrolls. Thus $X \cap \Sigma$ is a Cartier divisor on Σ which contains E. By [6, corollary 2.7] the family of residual curves to $E \subset X \cap \Sigma$ is a flat family of Cohen-Macaulay curves. The residual divisor $C \subset \Sigma$ is a connected, arithmetic genus 0 curve of degree 4 contained in X, i.e. a point of $\overline{H}^{4,0}$. So we have 2 projection morphisms $p_1 : N \to \overline{H}^{5,1}$ and $p_2 : N \to \overline{H}^{4,0}$.

One thing to notice is that if E is a quintic elliptic curve whose span is all of $\mathbb{P}^4$, then E is not contained in any hyperplanes, any quadric surfaces, or in any cone over a twisted cubic. Thus for every point $(\Sigma, E) \in N$, we have that Σ is a smooth cubic scroll.

Now define N^2 to be the fiber product $N \times_{\overline{H}^{5,1}} N$. We define the two maps $i = 1, 2$, $p_{2,i} : N^2 \to \overline{H}^{4,0}$ to be the compositions of the projection $p_i : N^2 \to N$ with $p_2 : N \to \overline{H}^{4,0}$.

Lemma 8.1. *For a general quartic rational curve $[C] \in \mathcal{H}^{4,0}(X)$, the fiber of $p_{2,i} : N^2 \to \overline{H}^{4,0}$ is unirational of dimension 4.*

Proof. Let $C \subset X$ be a general quartic rational curve. Let $T \subset \mathbb{P}^4$ be the threefold swept out by all 2-secant lines and tangent lines to C. Then $T \cap X$ is a surface. Let $U \subset X$ denote the complement of T. We construct a rational transformation $\rho : C \times U \to N^2$ as follows. For each point $p \in C$ and $q \in U$, there is a line $L = \mathrm{span}(p, q)$. The quotient projective space $\mathbb{P}^4/L$ is isomorphic to $\mathbb{P}^2$. Since $L \cap C = \{p\}$, we see that the image of C under the rational projection $\mathbb{P}^4 \to \mathbb{P}^2$ is a singular cubic curve. Let $V \subset C \times U$ denote the open set such that this singular cubic curve is a nodal cubic. Then the node corresponds to the unique 2-secant line $D \subset \mathbb{P}^4$ to C which intersects L. And there is a unique isomorphism $\pi : C \to D$ such that $\pi(r) = r$ for each $r \in C \cap D$ and such that $\pi(p) = s$ where $D \cap L = \{s\}$. By subsection 5.2 the 4-tuple $(C, D, \pi, C \cap D)$ determines a cubic scroll $f : \Sigma \to \mathbb{P}^4$ which contains C and D. Let E denote the residual curve to C in f^*X. Then (Σ, E) is a point of $\widetilde{H}$. Also we know that q is on E. The linear system $|2q|$ on E is a g^1_2, i.e. a degree 2 morphism $\pi : E \rightarrow \mathbb{P}^1$. By lemma5.5 $\pi : E \rightarrow \mathbb{P}^1$ determines a second cubic scroll Σ' containing E such that $|2q|$ is just the linear system of intersections of E with the lines of ruling of Σ'. For $(p, q) \in V$ we define $\rho(p, q) = ((\Sigma, E), (\Sigma', E))$.

Since every g^1_2 on E can be expressed as $|2q|$ for some $q \in E$, it is clear that we can obtain every cubic scroll Σ' containing E simply by varying the point q. Thus we conclude that the map $\rho : V \rightarrow N^2$ dominates the fiber of $p_{2,i} : N^2 \rightarrow \overline{H}^{4,0}$ over $[C]$. Since V is an open subset of the product of unirational varieties $C \times X$, we conclude that V is unirational. Since the image of a unirational variety is unirational, we conclude that the fiber of $p_{2,i} : N^2 \rightarrow \overline{H}^{4,0}$ over $[C]$ is unirational. ☐

Let $P \subset N^2$ denote the irreducible component whose general member is a point (E, Σ_1, Σ_2) where E is a smooth quintic elliptic and Σ_1, Σ_2 are cubic scrolls. Now consider the morphism

$$r : P \rightarrow \overline{H}^{4,0} \times \overline{H}^{4,0}, r(E, \Sigma_1, \Sigma_2) = (p_2(E, \Sigma_1), p_2(E, \Sigma_2)) \tag{29}$$

i.e. the pair of residual quartic curves to E in $\Sigma_1 \cap X$ and $\Sigma_2 \cap X$.

Theorem 8.2. *For a general pair (C_1, C_2) of smooth quartic rational curves in the image of r, the fiber is 1-dimensional. And the general fiber of the Abel-Jacobi map*

$$u_{4,0} : \mathcal{H}^{4,0}(X) \rightarrow J(X) \tag{30}$$

it one of the irreducible, unirational 3-folds $p_{2,2}(p_{2,1}^{-1}([C]))$ for some $[C] \in \mathcal{H}^{4,0}(X)$.

Proof. Let $\Pi \subset \mathbb{P}^4$ be a hyperplane such that $Y = \Pi \cap X$ is a smooth cubic surface. For a general point $p \in Y$ we can find a quartic elliptic curve $B \subset Y$ with $p \in B$: in the model of Y as the blow-up of $\mathbb{P}^2$ at 6 points, B is the proper transform of a plane cubic passing through 5 of those points and the additional point p. Also for a general point $p \in Y$, we can find a line of type I, $L \subset X$, such that $L \cap \Pi = \{p\}$. Define $E = B \cup L$, so E is a connected, nodal curve of arithmetic genus 1 and degree 5.

Claim 8.3. *The curve E satisfies the conditions in* [6, lemma 2.3]*, i.e.*

$$H^1(L, N_{L/X}(-p)) = H^1(B, N_{B/X}) = 0. \tag{31}$$

Therefore $Hilb_{5t}(X)$ is smooth at $[E]$ and deformations of E smooth the node at p.

We have the following short exact sequences of coherent sheaves:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N_{B/X} & \longrightarrow & N_{E/X}|_B & \longrightarrow & \mathcal{O}_p & \longrightarrow & 0 \\ 0 & \longrightarrow & N_{L/X} & \longrightarrow & N_{E/X}|_L & \longrightarrow & \mathcal{O}_p & \longrightarrow & 0 \end{array} \tag{32}$$

We also have a short exact sequence:

$$0 \longrightarrow N_{B/Y} \longrightarrow N_{B/X} \longrightarrow N_{Y/X}|_B \longrightarrow 0 \tag{33}$$

The self-intersection of a quartic rational curve on a smooth cubic scroll is 4. And $N_{Y/X}|_B = \mathcal{O}_\Pi(1)|_E$ is also degree 4. So by Riemann-Roch we conclude that $H^1(B, N_{B/Y}) = H^1(B, N_{Y/X}|_B) = 0$. Applying the long exact sequence in cohomology to our last short exact sequence, we conclude that $H^1(B, N_{B/X}) = 0$. Applying the long exact sequence in cohomology to our first short exact sequence above, we conclude that $H^1(B, N_{E/X}|_B) = 0$.

Twisting the second exact sequence above by $\mathcal{O}_L(-p)$ yields an exact sequence:

$$0 \longrightarrow N_{L/X}(-p) \longrightarrow N_{E/X}|_L(-p) \longrightarrow \mathcal{O}_p \longrightarrow 0 \tag{34}$$

By assumption $N_{L/X} \cong \mathcal{O}_L \oplus \mathcal{O}_L$, thus $N_{L/X}(-p) \cong \mathcal{O}_L(-1) \oplus \mathcal{O}_L(-1)$. In particular, $H^1(L, N_{L/X}(-p)) = 0$. Applying the long exact sequence in cohomology to this short exact sequence, we conclude that $H^1(L, N_{E/X}(-p)) = 0$. Finally, we have the short exact sequence:

$$0 \longrightarrow N_{E/X}|_L(-p) \longrightarrow N_{E/X} \longrightarrow N_{E/X}|_B \longrightarrow 0 \tag{35}$$

Applying the long exact sequence in cohomology to this short exact sequence, we conclude that $H^1(E, N_{E/X}) = 0$. Thus $[E] \in \mathrm{Hilb}_{5t}(X)$ is unobstructed. This finishes the proof of claim 8.3.

For a general line $M \subset \Pi$ containing p, the residual to L in $\mathrm{span}(L, M) \cap X$ is a smooth conic. Let M_1, M_2 be two such lines. Without loss of generality, we may also suppose that M_1, M_2 are 2-secant lines to B: since we are free to choose B a general quartic elliptic, we may first choose M_1, M_2, and then choose B to pass through p and one of the other two points of $M_i \cap Y$ for each of $i = 1, 2$. Let $M_i \cap B = \{p, q_i\}$ and let r_i denote the third point of $M_i \cap Y$.

Define $S_i' = \mathrm{span}(L, M_i)$ and let $S_1'', S_2'' \subset \Pi$ be smooth quadric surfaces containing $B \cup M_1$ and $B \cup M_2$ respectively. Define $S_1 = S_1' \cup S_1'', S_2 = S_2' \cup S_2''$, thus S_1, S_2 are each a union of a 2-plane and a smooth quadric surface. By [6, lemma 6.2] such a surface is a specialization of a cubic scroll, thus $[S_1], [S_2] \in \mathrm{Hilb}_{P(t)}(\mathbb{P}^4)$.

Claim 8.4. *For $i = 1, 2$, we have $H^1(S_i, I_{E/S_i} N_{S_i/\mathbb{P}^4}) = H^2(S_i, I_{E/S_i} N_{S_i/\mathbb{P}^4}) = 0$ where I_{E/S_i} is the ideal sheaf of $E \subset S_i$.*

By the deformation theory argument in the proof of [6, lemma 6.2], this vanishing result implies that the morphism $N^2 \to \overline{H}^{5,1}$ is smooth at (E, S_1, S_2).

We have a short exact sequence:

$$0 \longrightarrow N_{S_i/\mathbb{P}^4}|_{S_i'}(-M_i - L) \longrightarrow I_{E/S_i} N_{S_i/\mathbb{P}^4} \longrightarrow N_{S_i/\mathbb{P}^4}|_{S_i''}(-B) \longrightarrow 0 \tag{36}$$

And we have the two short exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N_{S_i'/\mathbb{P}^4}(-M_i - L) & \longrightarrow & N_{S_i/\mathbb{P}^4}|_{S'}(-M_i - L) & \longrightarrow & N_{M_i/S_i''}(-p) & \longrightarrow & 0 \\ 0 & \longrightarrow & N_{S_i''/\mathbb{P}^4}(-B) & \longrightarrow & N_{S_i/\mathbb{P}^4}|_{S_i''}(-B) & \longrightarrow & N_{M_i/S'} \otimes N_{M_i/S_i''}(-p-q) & \longrightarrow & 0 \end{array} \tag{37}$$

From the proof of lemma [6, lemma 6.3], we know that

$$N_{S_i'/\mathbb{P}^4} \cong \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(1), N_{S_i''/\mathbb{P}^4} \cong \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1,1) \oplus \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(2,2). \tag{38}$$

Thus we have $N_{S'/\mathbb{P}^4}(-M_i - L) \cong \mathcal{O}_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(-1)$ and $N_{S_i''/\mathbb{P}^4}(-B) \cong \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-1,-1) \oplus \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}$. Moreover $N_{M_i/S'} \cong \mathcal{O}_{\mathbb{P}^1}(1)$ and $N_{M_i/S_i''} \cong \mathcal{O}_{\mathbb{P}^1}$. Thus our two exact sequences are:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(-1) & \longrightarrow & N_{S_i/\mathbb{P}^4}|_{S_i'}(-M_i - L) & \longrightarrow & \mathcal{O}_{\mathbb{P}^1}(-1) & \longrightarrow & 0 \\ 0 & \longrightarrow & \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-1,-1) \oplus \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1} & \longrightarrow & N_{S_i/\mathbb{P}^4}|_{S_i''}(-B) & \longrightarrow & \mathcal{O}_{\mathbb{P}^1}(-1) & \longrightarrow & 0 \end{array} \tag{39}$$

It quickly follows from the long exact sequence in cohomology that

$$H^{j>0}(S', N_{S_i/\mathbb{P}^4}|_{S'}(-M_i - L)) = H^{j>0}(S_i'', N_{S_i/\mathbb{P}^4}|_{S_i''}(-B)) = 0 \tag{40}$$

Applying the long exact sequence in cohomology to our first short exact sequence, we conclude that $H^{j>0}(S_i, I_{E/S_i} N_{S_i/\mathbb{P}^4})$ This proves claim 8.4

We conclude that $N^2 \to \overline{H}^{5,1}$ is smooth at (E, S_1, S_2). Since $[E]$ is a smooth point of $\mathrm{Hilb}_{5t}(X)$ which lies in $\overline{H}^{5,1}$, we conclude that N^2 is smooth at (E, S_1, S_2) and the irreducible component of N^2 which contains (E, S_1, S_2) dominates $\overline{H}^{5,1}$, i.e. the irreducible component is P.

Claim 8.5. *The fiber of $r : P \to \overline{H}^{4,0} \times \overline{H}^{4,0}$ containing (E, S_1, S_2) is one-dimensional at (E, S_1, S_2).*

Of course the residual to L in $S_i' \cap X$ is a smooth conic D_i' which intersects M_i in $\{q_i, r_i\}$. And the residual to B in $S_i'' \cap X$ is a smooth conic D_i'' which contains r_i. Let us define $D_i = D_i' \cup D_i''$. Then $r(E, S_1, S_2) = (D_1, D_2)$.

Now we want to determine the dimension of every irreducible component of the fiber $\Phi = r^{-1}(D_1, D_2)$ through (E, S_1, S_2). By lemma [6, lemma 6.3], we may restrict our attention to the open subset of Φ parametrizing pairs (C, R_1, R_2) such that each of R_1, R_2 is in $T \cup U$, i.e. R_i is either a cubic scroll or the union of a 2-plane and a quadric surface along a line. But for any conic G in a cubic scroll Σ with directrix D and fiber F, we know that $G \sim D + F$. Given a union of 2 such conics, G', G'', we see that $3H - G' - G'' \sim D + 4F$. Thus the residual to such a curve in $\Sigma \cap X$ would be a quintic curve of arithmetic genus 0, not arithmetic genus 1. We conclude that for (C, R_1, R_2) in Φ, we must have R_i is in T.

For an open neighborhood of (E, S_1, S_2) in Φ, we must have that D_i' lies in the irreducible component of R_i which is a 2-plane. Now suppose given such a (C, R_1, R_2). There is a unique 2-plane S_i' which contains D_i'.

Thus $R_i = S_i' \cup R_i''$ for some smooth quadric surface R_i''. And $C = L \cup A$ for some quartic elliptic A. Now $\mathrm{span}(A)$ contains both $\mathrm{span}(D_1'')$ and $\mathrm{span}(D_2'')$. Thus $\mathrm{span}(A) = \Pi$. Finally, since $S_i' \cap \Pi = M_i$, we can only have $S_i' \cap R_i'' = M_i$. In particular, we conclude that M_i intersects A in the points p, q_i. Thus $A \subset Y = \Pi \cap X$ is a quartic elliptic curve passing through p, q_1 and q_2. These points impose independent conditions on the 4-dimensional linear system of quartic elliptic curves residual to D_i''. Thus there is a 1-dimensional linear system of A's.

For each A in this pencil of quartic elliptic curves, we have $A \cup D_i'' \subset Y$ lies in $|\mathcal{O}_Y(2)|$. Since Y is linearly normal and lies in no quadric surfaces, we have $|\mathcal{O}_Y(2)| = |\mathcal{O}_\Pi(2)|$. Thus there is a unique quadric surface R_i'' containing $A \cup D_i''$. Since $\{p, q_i, r_i\} \subset M_i \cap R_i''$, we conclude that $M_i \subset R_i''$. Thus $R_i = S_i' \cup R_i''$ is a surface in T, and (A, R_1, R_2) is a point in the fiber Φ. So the fiber of Φ is one-dimensional at (E, S_1, S_2), which finishes the proof of claim 8.5.

By the above, Φ is a 1-dimensional irreducible variety in a neighborhood of (E, S_1, S_2). It follows by upper semicontinuity of the fiber dimension that the general fiber of $r : \mathbb{P} \to \overline{H}^{4,0}(X) \times \overline{H}^{4,0}(X)$ is at most 1-dimensional.

For a general rational quartic $[C] \in \mathcal{H}^{4,0}(X)$, it follows by lemma 8.1 that $p_{2,1}^{-1}([C]) \subset N^2$ is a unirational 4-fold. So the subvariety $Z_C := p_{2,2}(p_{2,1}^{-1}([C])$ is an irreducible, unirational variety. Also observe that the subvarieties Z_C sweep out $\mathcal{H}^{4,0}(X)$: given a curve $[D] \in \mathcal{H}^{4,0}(X)$, if we choose any curve $[C] \in Z_D$, then also $[D]$ is contained in Z_C.

Since the fiber dimension of $p_{2,i} : N^2 \to \mathcal{H}^{4,0,(}X)$ is at most 1, we conclude the dimension of Z_C is at least $4 - 1 = 3$. Since $J(X)$ contains no unirational subvarieties, Z_C is contained in a fiber of the Abel-Jacobi map $u_{4,0} : \mathcal{H}^{4,0,(}X) \to J(X)$. By theorem 6.4, the general fiber of $u_{4,0}$ is irreducible of dimension 3. Combining this with the fact that a general point of $\mathcal{H}^{4,0}(X)$ is contained in an irreducible 3-fold Z_C, we conclude that the general fiber of $u_{4,0}$ is one of the general subvarieties Z_C, and vice versa. As a consequence, observe that the general fiber dimension of $u_{4,0}$ is 1 (and not 0). This completes the proof of the theorem. □

Corollary 8.6. *The general fiber of the Abel-Jacobi map $u_{5,1} : \mathcal{H}^{5,1}(X) \to J(X)$ is an irreducible, unirational 5-fold.*

Proof. Recall from the proof of lemma 8.1 that not only did we show that $p_{2,i}^{-1}([C])$ is unirational, but we showed that there is a dominant, generically-finite morphism $B \to p_{2,i}^{-1}([C])$ such that B is unirational and on B we can produce a section σ of the family of elliptic curves E – in fact we used $|2\sigma|$ as the g_2^1's on E to produce the surface Σ'. Thus we have a distinguished line on each Σ' corresponding to $|2\sigma|$. And if C' is the residual to $E \subset \Sigma' \cap X$, this line intersects C' in a distinguished point. What this shows is that there is a unirational variety B dominating a general fiber $Z = u_{4,0}^{-1}(p)$ such that after we base-change to B, we have a section τ of the family of quartic rational curves C'. Thus the family of quartic rational curves over B is a conic bundle with a section; therefore it is a $\mathbb{P}^1$-bundle.

It is easy to see that given a quartic rational curve C' along with a point p, the locus of cubic scrolls containing C is canonically birational to $\mathrm{Sym}^2(C) \times \mathbb{C}^*$, which is canonically birational to $\mathbb{A}^3$. Thus the fiber product $\widetilde{H} \times_{\mathcal{H}^{4,0}(X)} B \to B$ is canonically birational to $\mathbb{A}^3 \times B \to B$. In particular, since B is itself unirational, we conclude that $\widetilde{H} \times_{\mathcal{H}^{4,0}(X)} B$ is also unirational.

Consider the composite morphism:

$$\widetilde{H} \times_{\mathcal{H}^{4,0}(X)} B \longrightarrow \widetilde{H} \longrightarrow \mathcal{H}^{5,1}(X). \tag{41}$$

Define Y to be the image. Because $\widetilde{H} \to \mathcal{H}^{5,1}(X)$ has fiber dimension, we conclude that

$$\dim(Y) \geq \dim \widetilde{H} \times_{\mathcal{H}^{4,0}(X)} B - 1 = \dim(B) + 3 - 1 = 3 + 3 - 1 = 5. \tag{42}$$

So Y is a unirational variety of dimension 5. Since $J(X)$ contains no unirational varieties, we conclude that Y is contained in a fiber of $u_{5,1}$. Since $\widetilde{H} \to \mathcal{H}^{5,1}(X)$ is dominant, we conclude that a general point of $\mathcal{H}^{5,1}(X)$ is contained in one of the varieties Y. Finally, by theorem 7.2, we see that the general fiber of $u_{5,1}$ is an irreducible 5-fold. Thus we have that Y equals a fiber of $u_{5,1}$, so the general fiber of $u_{5,1}$ is an irreducible, unirational 5-fold. □

9. Quintic Rational Curves

In this section we will prove that the Abel-Jacobi map $u_{5,0} : \mathcal{H}^{5,0}(X) \to J(X)$ is dominant and the general fiber is an irreducible, unirational 5-fold.

The construction we use to understand quintic rational curves is as follows. Define $I \subset \mathcal{H}^{5,0}(X) \times \mathbb{G}(1,4)$ to be the locally closed subvariety parametrizing pairs $([C],[L])$ where L is a 3-secant line to C which is not a 4-secant line. There is a unique isomorphism $\phi : C \to L$ which is the identity on $C \cap L$. By lemma 5.6, there is a cubic scroll $h : \Sigma \to \mathbb{P}^4$ such that L is the directrix of Σ, $C \subset \Sigma$, and ϕ is the restriction of the projection $\pi : \Sigma \to L$ to C. The residual to C in $\Sigma \cap X$ is a quartic rational curve C' (which does not intersect L and which intersects lines of the ruling in a degree 2 divisor).

Lemma 9.1. *The scheme $\mathcal{H}^{5,0}(X)$ is irreducible of dimension 10. The morphism $I \to \mathcal{H}^{5,0}(X)$ is birational. For a general pair $([C],[L])$ the residual quartic curve C' is a smooth, nondegenerate quartic rational curve.*

Proof. This follows from [6, corollary 9.3, theorem 9.4]. □

What are the fibers of $I \to \mathcal{H}^{4,0}(X)$? By lemma 5.1, the fiber over $[C']$ for C' a general quartic rational curve is simply an open subset of the $\mathbb{P}^2$ which parametrizes the collection of g^1_2's on C'.

Theorem 9.2. *The general fiber of $u_{5,0} : \mathcal{H}^{5,0,(}X) \to J(X)$ is an irreducible, unirational 5-fold.*

Proof. Since $I \to \mathcal{H}^{5,0}(X)$ is birational, we will show that the general fiber of $u_{5,0} : I \to J(X)$ is an irreducible, unirational 5-fold.

Now I surjects to an open subscheme of $\mathcal{H}^{4,0,(}X)$ with irreducible fibers isomorphic to open subsets of $\mathbb{P}^2$. By the residuation trick, we know that $u_{5,0}$ is the pointwise inverse (up to constant translation) of the composite map:

$$I \longrightarrow \mathcal{H}^{4,0}(X) \xrightarrow{u_{4,0}} J(X) \tag{43}$$

We know the general fiber of $u_{4,0}$ is an irreducible 3-fold. Thus we conclude that the general fiber of $I_1 \to J(X)$ is an irreducible 5-fold.

To see that the general fiber is unirational, we once again use the fact that for general $p \in J(X)$ there is a morphism $B \to u_{4,0}^{-1}(p)$ such that B is unirational and the base-change to B of the universal curve over $\mathcal{H}^{4,0}(X)$ admits a section. Thus the base-change of the universal curve is birational to $B \times \mathbb{A}^1$. So the "relative symmetric product" of the universal curve is birational to $B \times \mathbb{A}^2$. By lemma 9.1 $I \to \mathcal{H}^{4,0}(X)$ is isomorphic to an open subset of the relative second symmetric product of the universal curve. Thus the fiber product $B \times_{\mathcal{H}^{4,0}(X)} I$ is birational to $B \times \mathbb{A}^2$. Since B is unirational, so is $B \times \mathbb{A}^2$. And $B \times_{\mathcal{H}^{4,0}(X)} I$ dominates the fiber over $p \in J(X)$. Thus we conclude that the general fiber of $I \to J(X)$ is unirational. □

10. Quintic Curves of Genus 2

By Bézout's theorem, X cannot contain a plane curve of degree $d > 3$. Thus the next case after quintic elliptic curves is quintic curves of genus 2.

In this section we will show that the Abel-Jacobi map $u_{5,2} : \mathcal{H}^{5,2}(X) \to J(X)$ has image $\text{image}(u_{5,2}) = u_{1,0}(F)$. Moreover the general fiber is irreducible and rational of dimension 8. The construction we will use to prove this is as follows. Suppose that $C \subset X$ is a smooth quintic curve of genus 2. By Riemann-Roch, $h^0(C, \mathcal{O}_{\mathbb{P}^4}(1)|_C) = 4$, so C is contained in a hyperplane section $H \cap X$. Similarly, $h^0(C, \mathcal{O}_{\mathbb{P}^4}(2)|_C) = 9 < 10$, so C is contained in a quadric surface $Q \subset H$. The residual to C in $Q \cap X$ is a line $L \subset X$. Thus there is a morphism $f : \widetilde{\mathcal{H}^{5,1}}(X) \to F$ where $\widetilde{\mathcal{H}^{5,1}}(X)$ is the normalization of $\mathcal{H}^{5,1}(X)$.

Lemma 10.1. *The scheme $\mathcal{H}^{5,2}(X)$ is irreducible of dimension 10 and the morphism $f : \widetilde{\mathcal{H}^{5,2}}(X) \to F$ is dominant.*

Proof. This follows from the proof of [6, theorem 10.1]. □

What is the fiber of f? Given a line $L \subset X$, to specify the curve C, it suffices to specify the hyperplane H containing L and the quadric surface $Q \subset H$ containing L. The collection of pairs

$$\mathbb{P}Q^\vee = \{([L],[H]) \in F \times \mathbb{P}^{4\vee} : L \subset H\} \tag{44}$$

is a $\mathbb{P}^2$-bundle over F. And the collection of triples

$$I = \{([L],[H],[Q]) : L \subset Q \subset H\} \tag{45}$$

is a $\mathbb{P}^6$-bundle over $\mathbb{P}Q^\vee$. By a dimension count we conclude that the morphism $g : \widetilde{\mathcal{H}^{5,2}}(X) \to I$ maps birationally to an open subset of I. Thus the fibers of f are irreducible, rational varieties of dimension 8.

Corollary 10.2. *The Abel-Jacobi map $u_{5,2}$ has image a (translate of) an open subset of $-u_{1,0}(F)$, and the general fiber of $u_{5,2}$ is an irreducible, rational 8-fold.*

Proof. By the residuation trick we know $u_{5,2}$ is the pointwise inverse of $u_{1,0} \circ g$. And $u_{1,0}$ is an embedding. Since g is dominant, we conclude that image$(u_{5,2})$ is an open subset of $-u_{1,0}(X)$. Since the fibers of $u_{5,2}$ equal the fibers of g, we conclude the general fiber of $u_{5,2}$ is an irreducible, rational variety of dimension 8. ☐

10.1. **Irreducibility for $\mathcal{H}^{d,0}(X)$.** We have avoided using the following result in the previous section, but we present it here to mention one important corollary.

Theorem 10.3. *For each d the space $\mathcal{H}^{d,0}(X)$ is an irreducible, reduced, local complete intersection scheme of dimension $2d$. Moreover the general point of $\mathcal{H}^{d,0}(X)$ is an unobstructed curve.*

This is theorem 1 of [7]. We will not discuss the proof here, but we will prove a corollary that follows from theorem 10.3 and our analysis of $\mathcal{H}^{4,0}(X)$ and $\mathcal{H}^{5,0}(X)$. Let $\widetilde{\mathcal{H}^{d,0}}(X)$ denote the normalization of $\mathcal{H}^{d,0}(X)$.

Theorem 10.4. *For each $d \geq 4$, the Abel-Jacobi morphism $\alpha_{d,0} : \widetilde{\mathcal{H}^{d,0}}(X) \to J(X)$ is dominant and the general fiber is irreducible.*

Proof. Let $\mathcal{H}^d \to \overline{\mathcal{H}^{d,0}(X)}$ denote the normalization of the closure of $\mathcal{H}^{d,0}(X)$. We will actually show that $\alpha_{d,0} : \mathcal{H}^d \to J(X)$ is dominant and the general fiber is irreducible. Let us denote the Stein factorization of $\alpha_{d,0}$ as follows:

$$\mathcal{H}^d \xrightarrow{\beta_d} Z_d \xrightarrow{\gamma_d} J(X) \tag{46}$$

We need to prove that γ_d is an open immersion. By theorem 10.3, we know that Z_d is irreducible. We will prove by induction that there is a rational section $\epsilon_d : J(X) \to Z_d$. It then follows that γ_d is an open immersion. We have already established this result in case $d = 4$ or 5. Therefore suppose that $d \geq 6$ and suppose that the theorem has been proved for all integers less than d (and greater than 3).

Let $\mathcal{H}^{d-2,0}(X)_u \subset \mathcal{H}^{d-2,0}(X)$ denote the open subscheme of $\mathcal{H}^{d-2,0}(X)$ such that the corresponding curve is unobstructed. Let $\pi : \mathcal{C}^{d-2} \to \mathcal{H}^{d-2,0}(X)_u$ denote the universal curve. The points of $\mathcal{C}^{d-2}$ parametrize pairs $([C],p)$ where $[C] \in \overline{\mathcal{H}^{d-2,0}}(X)$ and $p \in C$. Let $L \subset X$ be a general line. Let $U \subset \mathcal{C}^{d-2}$ denote the open subscheme parametrizing pairs $([C],p)$ such that $p \in X \setminus L$. We have a family of 2-planes $P \subset U \times \mathbb{P}^4$ whose fiber over $([C],p)$ is span(L,p). Let $D \subset U \times X$ denote the intersection of $U \times X$ with P inside $U \times \mathbb{P}^4$. Let $D_1 \subset D$ denote the constant family $U \times L \subset U \times \mathbb{P}^4$. Let D_2 denote the residual family of conics. By [6, corollary 2.7], D_2 is flat over U. Let $V \subset U$ denote the open subscheme parametrizing pairs $([C],p)$ such that the corresponding conic C' is smooth and such that C intersects C' transversally at p and in no other points. Then over V the union $\mathcal{C}^{d-2} \cup D_2 \subset V \times X$ is a flat family of curves of degree d and arithmetic genus 0. So there is an induced morphism $f : V \to \mathrm{Hilb}_{dt+1}(X)$.

We will prove that $f(V) \subset \overline{\mathcal{H}^{d,0}}(X)$. In fact we will prove that every point in $f(V)$ satisfies the conditions in [6, lemma 2.3], namely $H^1(C', N_{C'/X}(-p)) = H^1(C, N_{C/X}) = 0$. It then follows that such a point is a smooth point of $\mathrm{Hilb}_{dt+1}(X)$ and also that the node smooths so that the point is in $\overline{\mathcal{H}^{d,0}}(X)$. Let $([C],p)$ be a point of V, and let C' be the corresponding conic. Let $B = C \cup C'$ denote the union. By construction B is a connected nodal curve. We need to prove that $H^1(B, N_{B/X}) = 0$.

We have the exact sequence:

$$0 \longrightarrow N_{B/X}|_{C'}(-p) \longrightarrow N_{B/X} \longrightarrow N_{B/X}|_C \longrightarrow 0. \tag{47}$$

And we have two exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N_{C'/X}(-p) & \longrightarrow & N_{B/X}|_{C'}(-p) & \longrightarrow & \mathcal{O}_p & \longrightarrow & 0 \\ 0 & \longrightarrow & N_{C/X} & \longrightarrow & N_{B/X}|_C & \longrightarrow & \mathcal{O}_p & \longrightarrow & 0 \end{array}. \tag{48}$$

We have seen that $N_{C'/X}$ is either $\mathcal{O}_{C'}(1) \oplus \mathcal{O}_{C'}(1)$ or $\mathcal{O}_{C'}(2) \oplus \mathcal{O}_{C'}$. Thus $H^1(C', N_{C'/X}(-p)) = 0$. By the long exact sequence in cohomology we conclude that $H^1(C', N_{B/X}|_{C'}(-p)) = 0$. By assumption $H^1(C, N_{C/X}) = 0$, therefore also $H^1(C, N_{B/X}|_C) = 0$. By the long exact sequence in cohomology we conclude that $H^1(B, N_{B/X}) = 0$, i.e. B is unobstructed.

We conclude that $f(V)$ is contained in the smooth locus of $\overline{\mathcal{H}^{d,0}}(X)$. Thus we can factor f as $g : V \to \mathcal{H}^d$. By additivity $\theta(B) = \theta(C) + \theta(C')$. And by residuation we have $\theta(C') = -\theta(L)$ (up to a fixed constant). Thus we conclude that the composite map:

$$V \xrightarrow{g} \mathcal{H}^d \xrightarrow{\alpha_{d,0}} J(X) \tag{49}$$

equals the pointwise inverse (up to a constant translation) of the composite:

$$V \longrightarrow \mathcal{H}^{d-2} \xrightarrow{\alpha_{d-2,0}} J(X). \tag{50}$$

Since $V \to \mathcal{H}^{d-2}$ has irreducible fibers, the Stein factorization of this composite is just the Stein factorization of $\alpha_{d-2,0}$ (or more precisely an open subscheme). By the induction assumption $\gamma_{d-2} : Z_{d-2} \to J(X)$ is an open immersion. By the universal property of the Stein factorization of $V \to J(X)$, there is an induced morphism $\epsilon_d : Z_{d-2} \to Z_d$. This is a rational section of $\gamma_d : Z_d \to J(X)$, which shows that γ_d is an open immersion. Therefore the morphism $\alpha_{d,0} : \mathcal{H}^d \to J(X)$ is dominant and the general fiber is irreducible, as was to be shown. □

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Department of Mathematics, Harvard University, Cambridge MA 02138
E-mail address: harris@math.harvard.edu

Department of Mathematics, University of Michigan, Ann Arbor, MI 48109
E-mail address: mikeroth@math.lsa.umich.edu

Department of Mathematics, Massachusetts Institute of Technology, Cambridge MA 02139
E-mail address: jstarr@math.mit.edu