Introduction

1. If $n > m$ pigeons are placed into m boxes, then there exists (at least) one box with at least two pigeons.

2. If $n > m$, then any function $f : [n] \to [m]$ is not injective.($[n] = \{1, 2, \ldots, n\}$).

3. If $m(k-1)+1$ pigeons are placed in m boxes, then there exists (at least) one box that contains at least k pigeons.

4. If the average of m numbers $x_1, x_2, \ldots, x_m$ is greater(less) than a, then at least one x_i is greater(less) than a.

Warmup Problems

5. Among any 13 Queen's students, there are two born in the same month.

6. Among all the people in Canada, there are two that have the same number of hairs.

7. There are 25 students in the Putnam class and the sum of their ages is 514 years. Show that there are 17 students such that the sum of their ages is at least 350 years.
Proof. The average age of the students in the Putnam class is $\frac{514}{25} = 20.56$. If we consider all the possible groups of 17 students in the team, the average sum of the ages in all these groups is $17 \cdot 20.56 = 349.52$. This is because there are $\binom{25}{17}$ such groups and when adding up the sum of ages in all these groups, each person's age is counted exactly $\binom{24}{16}$ times. Assuming all the ages are integers, it follows by the averaging principle(question 4) that there is a group of 17 students whose ages sum up to at least 350.

Number Theory Problems

8. Let A be any set of 20 distinct integers chosen from the arithmetic progression 1, 4, 7, ..., 100. Prove that there must be two distinct integers in A whose sum is 104.

9. Given any $n+1$ integers in $[2n]$, show that one of them is divisible by another.
Proof. Let $a_1, a_2, \ldots, a_{n+1}$ be any integers between 1 and $2n$. Write $a_i = 2^{b_i}(2c_i + 1)$ for each i between 1 and $n+1$. Since the a_i's are between 1 and $2n$, it follows that the

possible values for $2c_i+1$ are $1, 3, \dots, 2n-1$. Hence, there exist $1 \le i \ne j \le n+1$ such that $2c_i + 1 = 2c_j + 1$. Then a_i divides a_j or a_j divides a_i.

10. If $n+1$ distinct numbers are selected from the set $[2n]$, prove that there are at least two numbers that are relatively prime.
Proof. Consider the holes $\{2i-1, 2i\}$ for $1 \le i \le n$. If we select any $n+1$ numbers between 1 and $2n$, two of them will be in the same hole, thus we will have 2 consecutive numbers. (NOTE: Relatively prime means that the numbers have no factor in common, like 80 and 55. Consecutive numbers are always relatively prime.)

11. Among any n integer numbers, there are some whose sum is divisible by n.
Proof. Let $a_1, a_2, \dots, a_n$ be any integer numbers. Consider the sums $s_i = a_1 + \cdots + a_i$ for $1 \le i \le n$. If any of the s_i's is divisible by n, then we are done. Otherwise, the remainder obtained when dividing s_i by n is in $\{1, 2, \dots, n-1\}$ for each i between 1 and n. By the pigeonhole principle, there exist $1 \le i < j \le n$ such that $s_i \equiv s_j \pmod{n}$, i.e., two sums with the same remainder. The difference of the sums (which is itself a sum) must then have remainder zero, and so be divisible by n.

12. Show that there is a multiple of 2005 whose digits are only 0 and 1.
Proof. Consider the numbers $a_i = 11\dots1$, where the number of 1's is i. Then the sequence $a_1, a_2, \dots, a_{2006}$ will contain at least two numbers $a_i \ne a_j$ such that $a_i \equiv a_j \pmod{2005}$. Just like problem 11, we then subtract the smaller one from the larger, and get something involving only ones and zeros with remainder zero when divided by 2005, i.e., a multiple of 2005. Trying this computationally, the first time this happens is with two hundred and one ones – it has remainder 1, the same as a single one! So, the number N with two hundred ones followed by a single zero is divisible by 2005.

13. Any sequence of $mn+1$ real numbers contains either an increasing sequence of length $m+1$ or a decreasing sequence of length $n+1$.
Proof. Proof by contradiction. Let $a_1, a_2, \dots, a_{mn+1}$ be a sequence of $mn+1$ real numbers such that the length of the longest increasing subsequence is at most m and the length of the longest decreasing sequence is at most n.

For each i with $1 \le i \le mn+1$, define $l_i = (u_i, v_i)$ where u_i is the length of the longest increasing subsequence of $a_1, a_2, \dots, a_{mn+1}$ that starts with a_i and v_i is the length of the longest decreasing sequence of $a_1, a_2, \dots, a_{mn+1}$ that starts with v_i.

Now since $1 \le u_i \le m$ and $1 \le v_i \le n$ for each i with $1 \le i \le mn+1$, by the pigeonhole principle it follows that there exists an integer j between 1 and $mn+1$ such that $u_i = u_j$ and $v_i = v_j$. This will lead to a contradiction.

14. Let u be an irrational real number. Let S be the set of all real numbers of the form $a + bu$, where a and b are integers. Show that S is dense in the real numbers, i.e., for

any real number x, and any $\epsilon > 0$, there is any element $y \in S$ such that $|x - y| < \epsilon$. (HINT: first try $x = 0$.)
Proof. Denote by $\{x\}$ the fractional part of x. For example, $\{1.25\} = 0.25$ and $\{-2.35\} = 0.65$. We solve the problem for $\epsilon \in (0, 1)$ and $x = 0$. Let N be an integer number such that $N > \frac{1}{\epsilon}$. Consider the numbers(*pigeons*) $\{0 \cdot u\}, \{1 \cdot u\}, \{2\dot{u}\}, \ldots, \{N \cdot u\}$ and the intervals(*holes*) $[0, \frac{1}{N}), [\frac{1}{N}, \frac{2}{N}), \ldots, [\frac{N-1}{N}, 1)$.

By the pigeonhole principle, there exist p and q between 0 and N such that $|\{pu\} - \{qu\}| < \frac{1}{N}$. Hence, $|(p-q)u + a| < \frac{1}{N}$, where $a = [qu] - [pu] \in \mathbb{Z}$. Let $b = p - q$. Then $|a + bu| < \frac{1}{N} < \epsilon$.

15. For every n in the set $\mathbb{Z}^+ = \{1, 2, \ldots\}$ of positive integers, let r_n be the minimum value of $|c - d\sqrt{3}|$ for all nonnegative integers c and d with $c + d = n$. Find, with proof, the smallest positive real number g with $r_n \leq g$ for all $n \in \mathbb{Z}$.

16. Prove that there is some power of 2 that begins 2005….
Proof. Talking about the first digits of numbers is the same as talking about the fractional parts of the base 10 logarithm of the number (the integer part of the logarithm determines the power of 10 to multiply by, and the fractional part the digits that appear). Look at the interval I of numbers x in $(0, 1)$ such that 10^x begins $0.2005\ldots$. If $u = \log_{10} 2$, then saying that 2^n begins $2005\ldots$ is the same as saying that nu is equal to a positive integer m plus some number in I. Since (by question 14) we know that numbers of the form $nu - m$ are dense in $\mathbb{R}$, there must be many of them in I, and therefore many n's so that 2^n begins with the digits $2005\ldots$.

Combinatorics Problems

17. Show that if there are n people at a party, then two of them know the same number of people (among those present).
Proof. If there is a person that knows $n - 1$ people at the party, then every person knows at least one other person at the party. Hence, we have n people and each of them knows at least one and at most $n - 1$ other people. By the pigeonhole principle, two persons know the same number of people.

If each person knows at most $n - 2$ people at the party, then apply the pigeonhole principle again.

18. Prove that in any group of six people there are either three mutual friends or three mutual strangers. (HINT: Represent the people by the vertices of a regular hexagon. Connect two vertices with a red line segment if the couple represented by these vertices are friends; otherwise, connect them with a blue line segment. Consider one of the vertices, say v. At least three line segments emanating from v have the same colour. There are two cases to consider.)

19. HARDER: Find some n so that in any group of n people there are either four mutual friends or four mutual strangers.

20. If $\mathcal{F}$ is a family of subsets of $[n]$ such that $A \cap B \neq \emptyset$ for any $A, B \in \mathcal{F}$, then $|\mathcal{F}| \leq 2^{n-1}$. Find a family $\mathcal{F}$ of 2^{n-1} subsets that satisfies the previous property.
Proof. Proof by contradiction. Assume that there exists a family $\mathcal{F}$ of subsets of $[n]$ such that $A \cap B \neq \emptyset$ for any $A, B \in \mathcal{F}$ and $|\mathcal{F}| \geq 2^{n-1} + 1$.

Consider the holes $h_A = \{A, [n] \setminus A\}$ with $A \subset [n]$. The number of distinct holes is 2^{n-1}. This is because we have 2^n subsets A of $[n]$ and $h_A = h_{[n]\setminus A}$ for each $A \subset [n]$.

Hence, we have at least $2^{n-1} + 1$ subsets in $\mathcal{F}$ and 2^{n-1} holes h_A. This will lead to a contradiction.

The family $\mathcal{F}_i = \{A \subset [n] : i \in A\}$ has the property that any two subsets in it are intersecting and its size is 2^{n-1}.

Geometry Problems

21. Five points lie in an equilateral triangle of side length 1. Show that two of the points lie no further than 1/2 apart. Can the "1/2" be replaced by anything smaller? Can it be improved if the "five" is replaced by "four"?
Proof. The midpoints of the sides of the triangle partition its surface into 4 equilateral triangles whose side length is $\frac{1}{2}$. From our 5 points, two will lie in the interior of one of this triangles.

To see that 5 is best possible, consider the three vertices of the triangle plus the center of the circle passing through those three points. The pairwise distances are 1 or $\frac{1}{\sqrt{3}} > \frac{1}{2}$.

22. The points of an infinite rectangular grid are colored with two colours. Show that there are two horizontal and two vertical lines with points at their intersection coloured with the same colour.

23. A lattice point in the plane is a point (x, y) such that both x and y are integers. Find the smallest number n such that given n lattice points in the plane, there exist two whose midpoint is also a lattice point.
Proof.

The midpoints of two lattice points (x_1, y_1) and (x_2, y_2) is a lattice point if and only if $x_1 + x_2$ and $y_1 + y_2$ are both even, i.e. x_1 and x_2 have the same parity and y_1 and y_2 have the same parity.

The possible parities of a lattice point coordinates are (even,even),(even,odd),(odd,even) and (odd,odd). By the pigeonhole principle, among any 5 lattice points, there will be two with coordinates of the same parity.

To see that 5 is best possible, consider the points $(0, 0), (0, 1), (1, 0)$ and $(1, 1)$.

24. A polygon in the plane has area 1.2432. Show that it contains two distinct points (x_1, y_1) and (x_2, y_2) which differ by (a, b) where a and b are integers.