# Problem Solving Practice Session

**The Rules.** There are way too many problems to consider in one session. Pick a few problems you like and play around with them. Don't spend time on a problem that you already know how to solve.

**The Hints.** Work in groups. Try small cases. Do examples. Look for patterns. Use lots of paper. Talk it over. Choose effective notation. Try the problem with different numbers. Work backwards. Argue by contradiction. Modify the problem. Generalize. Don't give up after five minutes. Don't be afraid of a little algebra. Sleep on it if need be. Ask.

## THE PROBLEMS

1. Prove that there are two people in Canada right now with the same number of hairs on their heads (not counting bald people!).
2. A lattice point in the plane is a point $(x, y)$ such that both $x$ and $y$ are integers. Find the smallest number $n$ such that given $n$ lattice points in the plane, there exist two whose midpoint is also a lattice point.
3. Prove that in any group of six people there are either three mutual friends or three mutual strangers. (HINT: Represent the people by the vertices of a regular hexagon. Connect two vertices with a red line segment if the couple represented by these vertices are friends; otherwise, connect them with a blue line segment. Consider one of the vertices, say $v$. At least three line segments emanating from $v$ have the same colour. There are two cases to consider.)
4. (Putnam 2002, A2) Prove that given any five points on a sphere, there exists a closed hemisphere which contains four of the points.
5. (Putnam 1990, A3) Prove that any convex pentagon whose vertices (no three of which are collinear) have integer coordinates must have area at least $5/2$.
6. Given any $n+1$ integers between $1$ and $2n$, show that one of them is divisible by another.
7. In a tournament between $20$ players, there are $14$ games (each between two players). Each player is in at least one game. Show that we can find $6$ games involving $12$ different players.
8. (Putnam 1996, A3) Suppose that each of $20$ students has made a choice of anywhere from $0$ to $6$ courses from a total of $6$ courses offered. Prove or disprove: there are $5$ students and $2$ courses such that all $5$ have chosen both courses or all $5$ have chosen neither course.
9. (Putnam 2000, B1) Let $a_j$, $b_j$, $c_j$ be integers for $1 \leq j \leq N$. Assume, for each $j$, at least one of $a_j, b_j, c_j$ is odd. Show that there exists integers $r, s, t$ such that $ra_j + sb_j + tc_j$ is odd for at least $4N/7$ values of $j$, $1 \leq j \leq N$.
10. (Putnam 2000, B6) Let $B$ be a set of more than $2^{n+1}/n$ distinct points with coordinates of the form $(\pm 1, \pm 1, \ldots, \pm 1)$ in $n$-dimensional space with $n \geq 3$. Show that there are three distinct points in $B$ which are the vertices of an equilateral triangle.