Problem Solving Practice Session

The Rules. There are way too many problems to consider in one session. Pick a few problems you like and play around with them. Don't spend time on a problem that you already know how to solve.

The Hints. Work in groups. Try small cases. Do examples. Look for patterns. Use lots of paper. Talk it over. Choose effective notation. Try the problem with different numbers. Work backwards. Argue by contradiction. Modify the problem. Generalize. Don't give up after five minutes. Don't be afraid of a little algebra. Sleep on it if need be. Ask.

THE PROBLEMS

1. Show that the equation $a^2 + b^2 = 20071203$ doesn't have an integer solution. Show that the equation $a^2 + b^2 + c^2 = 1983095288359$ doesn't have an integer solution.

2. (a) Find the last two digits of 2^{2007}, 3^{2007}, and 6^{2007}. (b) How would you find the *first* two digits of the same numbers? You can use a calculator if needed.

3. Prove that there is a multiple of 3^{2007} that
   (a) ends in 2007.
   (b) contains all the digits 0, 1, ..., 9 at least once.
   (c) contains only the digits 0 and 1.

4. Suppose that the polynomial $P(x)$ has integer coefficients, and that none of the integers $P(1)$, $P(2)$,..., $P(2007)$ is divisible by 2007. Prove that $P(x)$ has no integer roots.

5. Show that if n divides one Fibonacci number then it will divide infinitely many of them. (The Fibonacci numbers $\{F_k\}$ are 1, 1, 2, 3, 5, 8,... with $F_k = F_{k-1} + F_{k-2}$.)

6. Let A be the sum of the decimal digits of 4444^{4444}, let B be the sum of the digits of A, and C the sum of the digits of B. Problem: find C.

7. Do there exist 2007 consecutive integers such that each is divisible by a perfect cube bigger than 1?

8. (Putnam A3, 1995) The decimal number $d_1 d_2 \dots d_9$ has nine (not necessarily distinct) decimal digits. The number $e_1 e_2 \dots e_9$ has the property that each of the nine 9-digit numbers formed by replacing just one of the digits d_i in $d_1 d_2 \dots d_9$ by the corresponding digit e_i ($1 \le i \le 9$) is divisible by 7. The number $f_1 f_2 \dots f_9$ is related to $e_1 e_2 \dots e_9$ in the same way: If we replace any digit e_i in $e_1 e_2 \dots e_9$ by f_i, the number is divisible by 7.

   Show that, for each i, $d_i - f_i$ is divisible by 7.

9. (Putnam 1994, B6) For any integer a, set $n_a = 101a - 100 \cdot 2^a$. Show that for $0 \le a, b, c, d \le 99$, $n_a + n_b \equiv n_c + n_d \mod 10100$ implies $\{a, b\} = \{c, d\}$.

Congruence Basics

- We write $b \mid a$ ("b divides a") iff there is an integer q with $a = qb$.

- We write $a \equiv b \mod n$ ("a is congruent to b mod n") if $n \mid (a - b)$, i.e., if both a and b have the same remainder when dividing by n. Saying $a \equiv 0 \mod n$ is the same thing as saying that $n \mid a$.

The relation $\equiv$ (congruence) works a lot like $=$ (equals):

- If $a \equiv b \mod n$ and $b \equiv c \mod n$ then $a \equiv c \mod n$.

- If $a \equiv b \mod n$ and $c \equiv d \mod n$ then $a + c \equiv b + d \mod n$ and $ac \equiv bd \mod n$.

- There exists an integer b such that $ab \equiv 1 \mod n$ iff $\gcd(a, n) = 1$. This b is the *inverse* of a mod n, and can be computed from the Euclidian Algorithm. If $n = p$ is a prime number, then $\gcd(a, p) = 1$ iff $a \not\equiv 0 \mod p$, i.e., anything not congruent to zero mod p has an inverse mod p.

- Euler's function $\phi(n)$ counts how many positive integers between 1 and n are relatively prime to n. Facts: If $\gcd(n_1, n_2) = 1$ then $\phi(n_1 n_2) = \phi(n_1)\phi(n_2)$. If $p_1, p_2, \ldots, p_k$ are the prime factors of n then $\phi(n) = n(1 - 1/p_1)(1 - 1/p_2) \cdots (1 - 1/p_k)$.

- Euler's theorem: $a^{\phi(n)} \equiv 1 \mod n$ if $\gcd(a, n) = 1$. Fermat: $a^p \equiv a \mod p$ if p is prime.

- For any integer a, $a^2 \equiv 0$ or $1 \mod 4$, and $a^2 \equiv 0$, 1, or $4 \mod 8$ (i.e., those are the only possibilities).

- For any positive integer n, the sum of the digits of n is congruent to n mod 9 (this gives a good test for divisibility by 9 or 3). The alternating sum of the digits (starting with a positive sign from the bottom) is congruent to n mod 11.