# Problem Solving Practice Session

**The Rules.** There are way too many problems to consider in one session. Pick a few problems you like and play around with them. Don't spend time on a problem that you already know how to solve.

**The Hints.** Work in groups. Try small cases. Do examples. Look for patterns. Use lots of paper. Talk it over. Choose effective notation. Try the problem with different numbers. Work backwards. Argue by contradiction. Modify the problem. Generalize. Don't give up after five minutes. Don't be afraid of a little algebra. Sleep on it if need be. Ask.

## THE PROBLEMS

**1.** For $0 \leq m \leq k < n$, prove that $\binom{n}{k}\binom{k}{m} = \binom{n}{m}\binom{n-m}{k-m}$.

**2.** Show that $\sum_{k=1}^{n} k\binom{n}{k} = n2^{n-1}$, and that $\sum_{k=0}^{n} \frac{1}{k+1}\binom{n}{k} = \frac{2^{n+1}-1}{n+1}$

**3.** For $n, k \geq 0$ and $p$ prime, prove that $\binom{pa}{pb} \equiv \binom{a}{b} \pmod{p}$.

**4.** (Putnam 2004, B2) Let $m$ and $n$ be positive integers. Show that

$$\frac{(m+n)!}{(m+n)^{m+n}} < \frac{m!}{m^m}\frac{n!}{n^n}.$$

**5.** (Putnam 1992, B2) For nonnegative integers $n$ and $k$, define $Q(n,k)$ to be the coefficient of $x^k$ in the expansion of $(1+x+x^2+x^3)^n$. Prove that

$$Q(n,k) = \sum_{j=0}^{k} \binom{n}{j}\binom{n}{k-2j}.$$

**6.** (Putnam 2000, B2) Prove that $\dfrac{\gcd(m,n)}{n}\dbinom{n}{m}$ is an integer for all pairs of integers $n \geq m \geq 1$.

**7.** (Putnam 1991, B4) Suppose that $p$ is an odd prime. Prove that

$$\sum_{j=0}^{p} \binom{p}{j}\binom{p+j}{j} \equiv 2^p + 1 \pmod{p^2}.$$

**8.** (Putnam 1996, A5) If $p$ is a prime number greater than 3 and $k = \lfloor 2p/3 \rfloor$, prove that the sum

$$\binom{p}{1} + \binom{p}{2} + \cdots + \binom{p}{k}$$

of binomial coefficients is divisible by $p^2$.

**9.** If $n \equiv 0 \pmod{6}$, then calculate the number of subsets of $[n]$ whose size is congruent to $r \pmod{3}$ for each $r \in \{0,1,2\}$.

## SOME BASIC COMBINATORICS IDENTIES

**1.** The number of subsets of $[n] = \{1, 2, \ldots, n\}$ is $2^n$.

**2.** The number of permutations of $[n]$ is $n!$.

**3.** The *binomial coefficient* $\binom{n}{k}$ is defined to be the number of $k$-element subsets of $[n]$. Then $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

**4.** $\binom{n}{k} = \binom{n}{n-k}, k\binom{n}{k} = n\binom{n-1}{k-1}, \binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$

**5.** (**Newton's Binomial formula**) $(1+t)^n = \sum_{k=0}^{n} \binom{n}{k} t^k$

**6.** (**Vandermonde convolution formula**) $\sum_{i=0}^{k} \binom{m}{i}\binom{n}{k-i} = \binom{m+n}{k}$

**7.** $\sum_{k=0}^{n} \binom{n}{k} = 2^n, \sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$

**8.** The number of choices of $k$ objects from $n$ with repetitions allowed and order not significant is equal to the number of ways of choosing $n$ non-negative integers whose sum is $k$.

**9.** The number of $n$-tuples of non-negative integers $x_1, x_2, \ldots, x_n$ with $x_1 + x_2 + \cdots + x_n = k$ is $\binom{n+k-1}{n-1}$.

**10.** (**De Moivre formula**) $(\cos x + i \sin x)^n = \cos(nx) + i \sin(nx)$

**11.** If $p$ is a prime number, and $k$ the largest exponent so that $p^k$ divides $n!$, then

$$k = \left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor + \left\lfloor \frac{n}{p^3} \right\rfloor + \cdots$$

Another way to compute $k$ is to write $n$ in base $p$, i.e., write $n = a_0 + a_1 p + a_2 p^2 + \cdots$ with $0 \le a_i \le p-1$. Then $k = \dfrac{n - \sum a_i}{p-1}$.

**12.** (**Lucas Theorem**) Let $p$ be a prime and let $m = a_0 + a_1 p + \cdots + a_k p^k, n = b_0 + b_1 p + \cdots + b_k p^k$, where $0 \le a_i, b_i \le p-1$ for $i = 0, 1, \ldots, k-1$ (i.e., write $m$ and $n$ in base $p$). Then

$$\binom{m}{n} \equiv \prod_{i=0}^{k} \binom{a_i}{b_i} \pmod{p}$$