Problem Solving Practice Session

The Rules. There are way too many problems to consider in one session. Pick a few problems you like and play around with them. Don't spend time on a problem that you already know how to solve.

The Hints. Work in groups. Try small cases. Do examples. Look for patterns. Use lots of paper. Talk it over. Choose effective notation. Try the problem with different numbers. Work backwards. Argue by contradiction. Modify the problem. Generalize. Don't give up after five minutes. Don't be afraid of a little algebra. Sleep on it if need be. Ask.

THE PROBLEMS

1. (Putnam 2001, B1) Find a nonzero polynomial $P(x, y)$ such that $P(\lfloor a \rfloor, \lfloor 2a \rfloor) = 0$ for all real numbers a. Note: $\lfloor x \rfloor$ is the greatest integer less than or equal to x.

2. (Putnam 2002, A1) Let k be a fixed positive integer. The n-th derivative of $\frac{1}{x^k-1}$ has the form $\frac{P_n(x)}{(x^k-1)^{n+1}}$ where $P_n(x)$ is a polynomial. Find $P_n(1)$.

3. (Putnam 2003, B1) Do there exist polynomials $a(x)$, $b(x)$, $c(y)$, $d(y)$ such that

$$1 + xy + x^2y^2 = a(x)c(y) + b(x)d(y)\,?$$

4. (Putnam 1992, B4) Let $p(x)$ be a nonzero polynomial of degree less than 2007 having no nonconstant factor in common with $x^3 - x$. Let

$$\frac{d^{2007}}{dx^{2007}} \frac{p(x)}{x^3 - x} = \frac{f(x)}{g(x)}$$

for polynomials $f(x)$ and $g(x)$. Find the smallest possible degree of $f(x)$.

5. (Putnam 2002, A4) For each integer m, consider the polynomial

$$P_m(x) = x^4 - (2m+4)x^2 + (m-2)^2.$$

For what values of m is $P_m(x)$ the product of two nonconstant polynomials with integer coefficients?

6. (Putnam 2004, A4) Show that for any positive integer n, there is an integer N such that the product $x_1x_2\cdots x_n$ can be expressed identically in the form

$$x_1x_2\cdots x_n = \sum_{i=1}^{N} c_i(a_{i,1}x_1 + a_{i,2}x_2 + \cdots + a_{i,n}x_n)^n$$

where the c_i are rational numbers and each $a_{i,j}$ is one of the numbers -1, 0, 1.

7. (Putnam 1999, A2) Let $p(x)$ be a polynomial that is nonnegative for all real x. Prove that, for some k, there are polynomials $f_1(x), \ldots, f_k(x)$ such that $p(x) = \sum_{j=1}^{k} \left(f_j(x)\right)^2$.