Problem Solving Practice Session

The Rules. There are way too many problems to consider in one session. Pick a few problems you like and play around with them. Don't spend time on a problem that you already know how to solve.

The Hints. Work in groups. Try small cases. Do examples. Look for patterns. Use lots of paper. Talk it over. Choose effective notation. Try the problem with different numbers. Work backwards. Argue by contradiction. Modify the problem. Generalize. Don't give up after five minutes. Don't be afraid of a little algebra. Sleep on it if need be. Ask.

THE PROBLEMS

1. (Putnam 1986, A1) Find, with explanation, the maximum value of $f(x) = x^3 - 3x$ on the set of all real numbers x satisfying $x^4 + 36 \leq 13x^2$.

2. (Putnam 1990, B1) Find all real-valued continuously differentiable functions f on the real line such that, for all x, $\big(f(x)\big)^2 = \int_0^x \big[\big(f(t)\big)^2 + \big(f'(t)\big)^2\big]\,dt + 2007$.

3. (Putnam 1991, B2) Let f and g be nonconstant, differentiable, real-valued functions on $\mathbb{R}$. Suppose that for each pair of real numbers x and y,

   $$f(x+y) = f(x)f(y) - g(x)g(y) \qquad\qquad g(x+y) = f(x)g(y) + g(x)f(y)$$

   If $f'(0) = 0$, prove that $\big(f(x)\big)^2 + \big(g(x)\big)^2 = 1$ for all x.

4. (Putnam 1998, A3) Let f be a real function on the real line with continuous third derivative. Prove that there exists a point a such that $f(a) \cdot f'(a) \cdot f''(a) \cdot f'''(a) \geq 0$.

5. (Putnam 1997, B2) Let f be a twice-differentiable real-valued function satisfying $f(x) + f''(x) = -xg(x)f'(x)$ where $g(x) \geq 0$ for all real x. Prove that $|f(x)|$ is bounded.

6. (Putnam 1987, A3) For all real x, the real-valued function $y = f(x)$ satisfies the differential equation $y'' - 2y' + y = 2e^x$.
   (a) If $f(x) > 0$ for all real x, must $f'(x) > 0$ for all real x? Explain.
   (b) If $f'(x) > 0$ for all real x, must $f(x) > 0$ for all real x? Explain.

7. (Putnam 1994, B3) Find the set of all real numbers k with the following property: for any positive differentiable function f that satisfies $f'(x) > f(x)$ for all x, there is some number N such that $f(x) > e^{kx}$ for all $x > N$.

8. (Putnam 1991, A5) Find the maximum value of $\int_0^y \sqrt{x^4 + (y-y^2)^2}\,dx$ on $[0,1]$.

9. (Putnam 1986, A5) Suppose $f_1(x), f_2(x), \ldots, f_n(x)$ are functions of n real variables $x = (x_1, \ldots, x_n)$ with continuous second-order partial derivatives everywhere on $\mathbb{R}^n$. Suppose further that there are constants $c_{i,j}$ such that $\frac{\partial f_i}{\partial x_j} - \frac{\partial f_j}{\partial x_i} = c_{i,j}$ fo all $1 \leq i \leq n$ and $1 \leq j \leq n$. Prove that there is a function $g(x)$ on $\mathbb{R}^n$ such that $f_i + \partial g/\partial x_i$ is linear for all $1 \leq i \leq n$.