Problem Solving Practice Session

The Rules. There are way too many problems to consider in one session. Pick a few problems you like and play around with them. Don't spend time on a problem that you already know how to solve.

The Hints. Work in groups. Try small cases. Do examples. Look for patterns. Use lots of paper. Talk it over. Choose effective notation. Try the problem with different numbers. Work backwards. Argue by contradiction. Modify the problem. Generalize. Don't give up after five minutes. Don't be afraid of a little algebra. Sleep on it if need be. Ask.

The Problems

1. (Putnam 1996, B1) Define a *selfish* set to be a set which has its own cardinality (number of elements) as an element. Find, with proof, the number of subsets of $\{1, 2, \ldots, n\}$ which are *minimal* selfish sets, that is, selfish sets none of whose proper subsets is selfish.
2. (Putnam 2001, B1) Let n be an even positive integer. Write the numbers $1, 2, \ldots, n^2$ in the squares of an $n \times n$ grid so that the k-th row, from left to right, is $(k-1)n+1$, $(k-1)n+2, \ldots, (k-1)n+n$. Color the squares of the grid so that half of the squares in each row and in each column are red and the other half are black (a checkerboard coloring is one possibility). Prove that for each coloring, the sum of the numbers on the red squares is equal to the sum of the numbers on the black squares.
3. (Putnam 1995, B1) For a partition π of $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, let $\pi(x)$ be the number of elements in the part containing x. Prove that for any two partitions π and π', there are two distinct numbers x and y in $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ such that $\pi(x) = \pi(y)$ and $\pi'(x) = \pi'(y)$.
4. (Putnam 2005, A2) Let $\mathbf{S} = \{(a, b) | a = 1, 2, \ldots, n, b = 1, 2, 3\}$. A *rook tour* of $\mathbf{S}$ is a polygonal path made up of line segments connecting points $p_1, p_2, \ldots, p_{3n}$ in sequence such that
   (a) $p_i \in \mathbf{S}$,
   (b) p_i and p_{i+1} are unit distance apart for $1 \leq i < 3n$,
   (c) for each $p \in \mathbf{S}$ there is unique i such that $p_i = p$.
   How many rook tours are there that begin at $(1, 1)$ and end at $(n, 1)$?
5. (Putnam 2002, A3) Let $n \geq 2$ be an integer and T_n be the number of nonempty subsets S of $\{1, 2, 3, \ldots, n\}$ with the property that the average of the elements of S is an integer. Prove that $T_n - n$ is always even.
6. (Putnam 2005, B4) For positive integers m and n, let $f(m, n)$ denote the number of n-tuples $(x_1, x_2, \ldots, x_n)$ of integers such that $|x_1| + |x_2| + \cdots + |x_n| \leq m$. Show that $f(m, n) = f(n, m)$.
7. (Putnam 2003, A6) For a set S of nonnegative integers, let $r_S(n)$ denote the number of ordered pairs (s_1, s_2) such that $s_1 \in S$, $s_2 \in S$ and $s_1 + s_2 = n$. Is it possible to partition the nonnegative integers into two sets A and B in such a way that $r_A(n) = r_B(n)$ for all n?