Problem Solving Practice Session

The Rules. There are way too many problems to consider in one session. Pick a few problems you like and play around with them. Don't spend time on a problem that you already know how to solve.

The Hints. Work in groups. Try small cases. Do examples. Look for patterns. Use lots of paper. Talk it over. Choose effective notation. Try the problem with different numbers. Work backwards. Argue by contradiction. Modify the problem. Generalize. Don't give up after five minutes. Don't be afraid of a little algebra. Sleep on it if need be. Ask.

THE PROBLEMS

1. Let a_1 be a nonnegative real number and, for $n > 1$, let $a_{n+1} = 1 + \frac{1}{1+a_n}$. Prove that the sequence $\{a_n\}$ converges and find the limit.

2. (Putnam 1990, A1) Let $T_0 = 2$, $T_1 = 3$, $T_2 = 6$, and for $n \geq 3$

$$T_n = (n+4)T_{n-1} - 4nT_{n-2} + (4n-8)T_{n-3}\,.$$

The first few terms are $2, 3, 6, 14, 40, 152, 784, 5168, 40576, 363392$. Find, with proof, a formula for T_n of the form $T_n = A_n + B_n$ where $\{A_n\}$ and $\{B_n\}$ are well-known sequences.

3. (Putnam 1993, A2) Let $(x_n)_{n\geq 0}$ be a sequence of nonzero real numbers such that $x_n^2 - x_{n-1}x_{n+1} = 1$ for $n = 1, 2, 3, \ldots$. Prove there exists a real number a such that $x_{n+1} = ax_n - x_{n-1}$ for all $n \geq 1$.

4. (Putnam 1985, A3) Let d be a real number. For each integer $m \geq 0$, define a sequence $\{a_m(j)\}$, $j = 0, 1, 2, \ldots$ by the condition

$$a_m(0) = \frac{d}{2^m}\,, \qquad \text{and} \qquad a_m(j+1) = \big(a_m(j)\big)^2 + 2a_m(j)\,, \qquad j \geq 0\,.$$

Evaluate $\lim\limits_{n\to\infty} a_n(n)$.

5. (Putnam 1994, A1) Suppose that $a_1, a_2, a_3, \ldots$ satisfies $0 < a_n \leq a_{2n} + a_{2n+1}$ for all $n \geq 1$. Prove that the series $\sum\limits_{n=1}^{\infty} a_n$ diverges.

6. (Putnam 1988, B4) Prove that if $\sum\limits_{n=1}^{\infty} a_n$ is a convergent series of positive real numbers, then so is $\sum\limits_{n=1}^{\infty} (a_n)^{n/(n+1)}$.

7. (Putnam 1988, A3) Determine, with proof, the set of real numbers x for which

$$\sum_{n=1}^{\infty} \left[\frac{1}{n}\csc\left(\frac{1}{n}\right) - 1\right]^x$$

converges.