

Problem Solving Practice Session

The Rules. There are way too many problems to consider in one session. Pick a few problems you like and play around with them. Don't spend time on a problem that you already know how to solve.

The Hints. Work in groups. Try small cases. Do examples. Look for patterns. Use lots of paper. Talk it over. Choose effective notation. Try the problem with different numbers. Work backwards. Argue by contradiction. Modify the problem. Generalize. Don't give up after five minutes. Don't be afraid of a little algebra. Sleep on it if need be. Ask.

A POTPOURRI OF PROBLEMS

1. (Putnam 1988, A1) Let R be the region consisting of the points (x, y) of the Cartesian plane satisfying both $|x| - |y| \leq 1$ and $|y| \leq 1$. Sketch the region R and find its area.
2. (Putnam 2000, A1) Let A be a positive real number. What are the possible values of $\sum_{j=0}^{\infty} x_j^2$, given that x_0, x_1, \dots are positive real numbers for which $\sum_{j=0}^{\infty} x_j = A$?
3. (Putnam 2001, A1) Consider a set S and a binary operation $*$, i.e., for each $a, b \in S$, $a * b \in S$. Assume that $(a * b) * a = b$ for all $a, b \in S$. Prove that $a * (b * a) = b$ for all $a, b \in S$.
4. (Putnam 1998, A1) A right circular cone has base of radius 1 and height 3. A cube is inscribed in the cone so that one face of the cube is contained in the base of the cone. What is the side-length of the cube?
5. (Putnam 1989, A1) How many primes among the positive integers, written as usual in base 10, are such that their digits are alternating 1's and 0's beginning and ending with 1?
6. (Putnam 1996, A1) Find the least number A such that for any two squares of combined area 1, a rectangle of area A exists such that the two squares can be packed in the rectangle (without the interiors of the squares overlapping). You may assume that the sides of the squares will be parallel to the sides of the rectangle.
7. (Putnam 1989, B1) A dart, thrown at random, hits a square target. Assuming that any two parts of the target of equal area are equally likely to be hit, find the probability that the point hit is nearer to the center than to any edge. Express your answer in the form $(a\sqrt{b} + c)/d$ where a, b, c, d are positive integers.
8. (Putnam 1998, B1) Find the minimum value of

$$\frac{(x + 1/x)^6 - (x^6 + 1/x^6) - 2}{(x + 1/x)^3 + (x^3 + 1/x^3)}$$

for $x > 0$.