Math 419/819 **Final Exam** DECEMBER 2003

INSTRUCTIONS: The exam has seven sets of questions labeled **I** through **VII**.

If you are registered in Math 419, choose and answer only *one* of these sets. If you are registered in Math 819, choose and answer any *two* of them.

You may freely use any notes or statements from class. It is also permissible to use outside sources, or to talk with other students about the exam questions, but if you do this be sure to acknowledge these discussions or references in your write up of those questions.

It is apparently mandatory to include the following statement:

> PLEASE NOTE: The candidate is urged to submit with the answer paper a clear statement of any assumptions made if doubt exists as to the interpretation of any question that requires a written answer.

Another possible solution is just to ask me about the question.

The exam is due on or before **Wednesday, December 17, 2003.**

Questions:

- I. — SUMS OF POINTS ON AN ELLIPTIC CURVE.
- II. — DESSINS D'ENFANT.
- III. — TWO CLASSICAL FACTS ABOUT THE DUAL CURVE.
- IV. — QUOTIENTS OF AFFINE CURVES BY FINITE GROUPS.
- V. — A DESCRIPTION OF THE PICARD GROUP BY GENERATORS AND RELATIONS.
- VI. — THE GENUS OF A CURVE ON $\mathbb{P}^1 \times \mathbb{P}^1$.
- VII. — LINE BUNDLES ON $\mathbb{P}^2$.

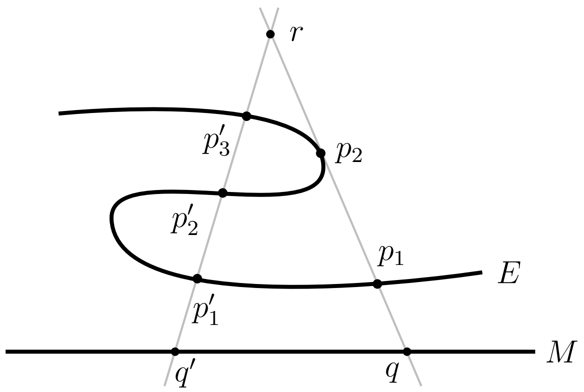
I. — SUMS OF POINTS ON AN ELLIPTIC CURVE.

INTRODUCTION

An elliptic curve is a curve E of the form $\mathbb{C}/\Lambda$ for some lattice $\Lambda \subset \mathbb{C}$. It is a genus 1 curve, and from the quotient description, we see that an elliptic curve is also automatically a group.

The theory of the Weierstrass $\wp$-function shows that any elliptic curve E can be embedded in $\mathbb{P}^2$ as a smooth cubic curve. In class we saw how to recover the group law (the one given by addition in $\mathbb{C}$ mod Λ) by using lines in $\mathbb{P}^2$ and their intersections with E.

That argument depended on one assertion which we didn't verify. Suppose that we fix a point $r \in \mathbb{P}^2$, and an auxiliary line M not passing through r. For any point q of M there is a unique line L_{rq} connecting r to q. This line will intersect E in three points, counting multiplicity:



In the picture, the line L_{rq} meets E in p_1 and p_2, but meets with multiplicity 2 at p_2. The line $L_{rq'}$ meets E in p_1', p_2', and p_3' each with multiplicity 1.

Suppose we add these points of E (again adding with multiplicity) in the group law of E. In the example above, this means that we would either add $p_1' + p_2' + p_3'$ if we picked $q' \in M$, or $p_1 + p_2 + p_2$ $(= p_1 + 2p_2)$ if we picked $q \in M$.

The result of the addition will be a single point of E, and so this procedure gives a map (of sets) $M \longrightarrow E$. The unverified assertion was that this was also a map of curves or Riemann surfaces, i.e., that it was locally described by holomorphic functions.

PURPOSE

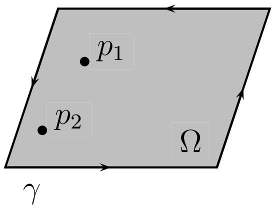
The point of this question is to justify this assertion, and also to give an example of *how* you can go about verifying statements of this type.

SETUP

Fix a point q of M and pick a chart U of $M \cong \mathbb{P}^1$ around q. Choose a local coordinate t on U so that, as usual, $t = 0$ is exactly the point q. We want to write the map $U \longrightarrow E$ as a function of t, and then see that it is a holomorphic function. Since we're interested in seeing if the map is holomorphic at q, we may restrict ourselves to the case that t is small.

The fact that $M \cong \mathbb{P}^1$, embedded in $\mathbb{P}^2$ as a line means that we can make a convenient (although not necessary) simplifying assumption: there are linear equations F_0 and F_1 in X, Y, and Z, so that $F_0 = 0$ is the line L_{rq} connecting r to q, and that in general for any value of t, $F_0 + tF_1 = 0$ is the equation of the line connecting r to the point of U corresponding to that value of t.

Now let $\Omega \subset \mathbb{C}$ be a fundamental domain for Λ and $\phi : \Omega \longrightarrow E$ the surjective map onto E. The line L_{rq} hits E in some points, which correspond to points in Ω. We can choose Ω such that these points are in the interior of Ω.



If we pull back the equations F_0 and F_1 using ϕ, we can translate the problem as follows. Given two holomorphic functions f_0 and f_1 on Ω, and any (small) value of $t \in \mathbb{C}$, consider the holomorphic function $f_0 + tf_1$ on Ω. Find the zeros of this function in Ω and add them (taking into account multiplicity of zeros). We need to show that this sum is a holomorphic function of t.

PROBLEMS

1. Suppose that $r = [0 : 1 : 0]$, M is the line $Y = 0$, $q = [1 : 0 : 0]$ and that for $t \in \mathbb{C}$ we parameterize nearby points q_t by $q_t = [1 : 0 : t] \in M$. Show that there exist linear equations F_0 and F_1 such that for any t, $F_0 + tF_1 = 0$ is the equation of the line connecting r to q_t.

2. Suppose that Ω is the fundamental domain as above, with coordinate z, and that γ is the boundary of Ω, oriented counterclockwise. If $f(z)$ is a holomorphic function on Ω, with zeros in the interior of Ω, explain why the integral

$$\frac{1}{2\pi i} \int_\gamma \frac{zf'(z)}{f(z)}\ dz$$

computes the sum of the zeros of $f(z)$ in Ω, with multiplicity.

3. We have now reduced the problem to the following: given holomorphic functions $f_0(z)$ and $f_1(z)$ on Ω such that the zeros of f_0 are in the interior of Ω, show that the following function of t:

$$S(t) := \frac{1}{2\pi i} \int_\gamma \frac{z(f_0'(z) + tf_1'(z))}{f_0(z) + tf_1(z)}\ dz$$

is a holomorphic function of t. We only need to show this for small values of t, and we use this restriction to guarantee that the zeros of $f_0(z) + tf_1(z)$ never lie on the boundary γ of Ω.

At this point you may choose one of two ways to show this (or make up your own). Both are based on the fact that, since we're integrating over a compact domain, we can interchange, in the first case, integration with respect to z and differentiation with respect to t, and in the second case, integration and infinite summation.

The first method is this. Write t as a sum $t = t_1 + it_2$ of imaginary and real parts. One way to say that $S(t)$ is a holomorphic function of t is to say that the Cauchy-Riemann operator $\frac{\partial}{\partial t_1} + i\frac{\partial}{\partial t_2}$ annihilates $S(t)$, i.e. that

$$\left(\frac{\partial}{\partial t_1} + i\frac{\partial}{\partial t_2}\right) S(t) = 0.$$

Interchanging differentiation and integration this is the same as saying that

$$\frac{1}{2\pi i} \int_\gamma \left(\frac{\partial}{\partial t_1} + i\frac{\partial}{\partial t_2}\right) \frac{z(f_0'(z) + tf_1'(z))}{f_0(z) + tf_1(z)}\ dz = 0,$$

and for this to happen, it is enough that for each fixed $z \in \gamma$,

$$\left(\frac{\partial}{\partial t_1} + i\frac{\partial}{\partial t_2}\right) \frac{z(f_0'(z) + tf_1'(z))}{f_0(z) + tf_1(z)} = 0.$$

Or, using once more what it means to be annihilated by the Cauchy-Riemann operator, it is enough to show that, for each fixed $z \in \gamma$ the function

$$\frac{z(f_0'(z) + tf_1'(z))}{f_0(z) + tf_1(z)}$$

is a holomorphic function of t. Explain why this is true, and so that (reversing the chain of implications above) $S(t)$ is a holomorphic function of t.

Alternatively, argue that along γ you can expand the function

$$\frac{z(f_0'(z) + tf_1'(z))}{f_0(z) + tf_1(z)}$$

as a power series in t with coefficients functions of z. Integrating around γ then gives a power series expression for $S(t)$, showing that it is holomorphic in t.

II. — DESSINS D'ENFANT.

INTRODUCTION.

There is a beautiful area of modern research concerned with trying to describe the Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ by using strange graphs scrawled on Riemann surfaces.

In this question there will be absolutely no explanation of this connection between these pictures and the Galois group, but you will get to draw some of the graphs.

Recall that for a map $\pi : C \longrightarrow D$ between curves or Riemann surfaces, a point q of D is called a *branch point* of π if it is the image of a ramification point. There may be fewer branch points than ramification points since different ramification points may map to the same point of D.

A map $\pi : C \longrightarrow \mathbb{P}^1$ is called a *Belyĭ Map* if it has at most three branch points. Any three distinct points on $\mathbb{P}^1$ are equivalent to any other three distinct points under an automorphism of $\mathbb{P}^1$, so it is traditional to take the three possible branch points to be $0\ (= [0:1])$, $1\ (= [1:1])$ and $\infty\ (= [1:0])$ on $\mathbb{P}^1$, and we will always assume this to be the case.

Let

$$I := \Big\{ x \in \mathbb{R} \mid 0 \leq x \leq 1 \Big\} \subset \mathbb{R} \subset \mathbb{C} \subset \mathbb{P}^1$$

be the unit interval, considered as a subset of $\mathbb{P}^1$.

For any Belyĭ map $\pi : C \longrightarrow \mathbb{P}^1$, the subset $\pi^{-1}(I)$ is a connected graph on C. Its vertices are the inverse images of 0 and 1 in $\mathbb{P}^1$, and the edges the inverse images of the open interval $(0,1)$. It is usual to colour the vertices mapping to 0 black, and the ones mapping to 1 white. Each edge connects a black vertex to a white one. This graph is called the "dessin" associated to the map π.

The dessin is usually considered to be drawn on the curve C. In order to simplify this question, we will only draw the abstract graph, and forget how it is realized as a subset of C. (This is an important part of the dessin, so this is a substantial simplification.)

PURPOSE.

The purpose of this question is to use our topological picture of maps between curves to draw some of the graphs. Of course, the easiest examples to compute are those when $C = \mathbb{P}^1$, which is what we'll concentrate on.

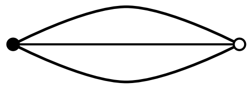
PROBLEMS

1. Let's first establish some general facts relating the topological data of the map to the data of the graph. Suppose that $\pi : C \longrightarrow \mathbb{P}^1$ is a Belyĭ map, and that $p \in C$ is a point of π mapping to either 0 or 1. The point p is not necessarily a ramification point

of π, but it is a vertex of the dessin. Show that the *valence* of this vertex is the same as the ramification index k_p of π at p. (The valence of a vertex of a graph is the number of edges connected to that vertex.)

2. If the Belyĭ map $\pi : C \longrightarrow \mathbb{P}^1$ is of degree d, explain why the associated dessin has exactly d edges. Also, justify the claim that each edge connects a white vertex to a black one. (This is why, in the definition of valence above, we didn't need to count "half-edges").

3. Suppose that E is an elliptic curve, and $\pi : E \longrightarrow \mathbb{P}^1$ a Belyĭ map of degree 3, with a single point $p_0 \in E$ mapping to 0 with ramification index $k_{p_0} = 3$, and a single point $p_1 \in E$ mapping to 1 with ramification index $k_{p_1} = 3$. Explain why the dessin associated to this map is given by the (abstract) graph below:



4. We now look at the case that the curve C is $\mathbb{P}^1$. Since both curves are now $\mathbb{P}^1$, we use the convention that $[\alpha : \beta]$ denotes a point on the image $\mathbb{P}^1$, and $[X : Y]$ denotes a point on the initial curve C ($\cong \mathbb{P}^1$).

Suppose that F_0 and F_1 are two homogeneous polynomials of degree d in X and Y without common factors. Explain why the map $\mathbb{P}^1 \cong C \longrightarrow \mathbb{P}^1$ given by:

$$\pi : [X : Y] \mapsto [F_0(X, Y) : F_1(X, Y)]$$

is well defined, that is, doesn't depend on the representative $[X : Y]$ chosen for a point $p \in \mathbb{P}^1$.

Also show that for any point $[\alpha : \beta] \in \mathbb{P}^1$, the inverse image $\pi^{-1}([\alpha : \beta])$ consists of roots $[X : Y] \in C$ of the polynomial $\beta F_0 - \alpha F_1$.

5. Since $\beta F_0 - \alpha F_1$ is a homogeneous polynomial of degree d in two variables, it factors into a product of linear factors:

$$\beta F_0 - \alpha F_1 = \prod_{s=1,\ldots,r} (b_s X_0 - a_s X_1)^{m_s},$$

possibly with multiplicities. Explain how the numbers m_s are connected with the ramification indices of the corresponding roots $[a_s : b_s]$ in $C \cong \mathbb{P}^1$, and prove your statement.

Using this method for understanding the ramification, check that the following maps are Belyĭ maps and work out the corresponding graph (Hint: to see that the only branch points are 0, 1, and ∞, it's probably easiest to use the method above to compute the ramification over 0, 1, and ∞, and then use Riemann-Hurwitz to verify that this accounts for all the possible ramification).

6. The map given by $F_0 = (X + 2Y)^3$, $F_1 = 3X^2Y + 9XY^2 + 7Y^3$.

7. The map given by $F_0 = 4X^3 - 12X^2Y - 15XY^2 - 4Y^3$, $F_1 = -27XY^2$.

8. The map given by $F_0 = 3^7X^4(9X^2 - 18XY + 25Y^2)$, $F_1 = -2^45^5Y^6$.

III. — TWO CLASSICAL FACTS ABOUT THE DUAL CURVE.

INTRODUCTION.

A line in $\mathbb{P}^2$ is described by an equation $aX + bY + cZ = 0$. If we simultaneously multiply the coefficients a, b, c by a common scalar $\lambda \in \mathbb{C}$, $\lambda \neq 0$, then the equation gets scaled by λ, but it's still the same line.

It follows that if we want to parameterize the lines in $\mathbb{P}^2$, we should look at the set

$$\{(a,b,c) \mid (a,b,c) \neq (0,0,0)\}/\sim$$

where the equivalence relation is that $(a,b,c) \sim (\lambda a, \lambda b, \lambda c)$ for any $\lambda \in \mathbb{C}^*$.

This quotient is exactly how we described $\mathbb{P}^2$. The quotient is called the "dual $\mathbb{P}^2$", and denoted $\mathbb{P}^{2^*}$ to distinguish it from the "actual" $\mathbb{P}^2$. A point $[a : b : c] \in \mathbb{P}^{2^*}$ is thought of as describing the line $aX + bY + cZ = 0$ in the "usual" $\mathbb{P}^2$.

The relationship is entirely reciprocal. If we fix a point $[X_0 : Y_0 : Z_0]$ in $\mathbb{P}^2$, the set of $[a : b : c]$ such that $aX_0 + bY_0 + cZ_0 = 0$ forms a line in $\mathbb{P}^{2^*}$. Thus each point of each of the $\mathbb{P}^2$'s parameterizes a line in the other.

Suppose we have a smooth curve C in $\mathbb{P}^2$. For every point $p \in C$, there is a unique tangent line L_p to C at p, and this corresponds to a unique point in $\mathbb{P}^{2^*}$. The process of sending a point p on C to the point of $\mathbb{P}^{2^*}$ corresponding to its tangent line gives a map $C \longrightarrow \mathbb{P}^{2^*}$. The image of this map is called the "dual curve", C^* of C.

(Thanks to question 5 on homework assignment 3, we can be a bit more explicit about this map. If the curve C corresponds to the homogeneous equation $F = 0$, then for any point $p \in C$, $[F_X(p) : F_Y(p) : F_Z(p)]$ is the image of p on the dual curve C^* in $\mathbb{P}^{2^*}$. This explicit description won't, unfortunately, be useful for us.)

PURPOSE

The purpose of this question is to compute two classical facts about the dual curve: the first is the degree of C^* (knowing the degree of C), and the second is that the dual of the dual is the original curve, i.e., that $(C^*)^* = C$, justifying the name.

PROBLEMS

1. To begin with we need to understand one more part of the relationship between $\mathbb{P}^2$ and $\mathbb{P}^{2^*}$. Let M be a line in $\mathbb{P}^{2^*}$, i.e., the set of points $[a : b : c] \in \mathbb{P}^{2^*}$ satisfying an equation of the form $aX_0 + bY_0 + cZ_0 = 0$ for some fixed X_0, Y_0, and Z_0.

Any point q on M corresponds to a line in $\mathbb{P}^2$. As we vary q on M we get a family of lines in $\mathbb{P}^2$. What does this collection of lines look like?

Show that there is a point r in $\mathbb{P}^2$ (depending on the line M) such that the lines L_q in $\mathbb{P}^2$ corresponding to points $q \in M$ are exactly the lines in $\mathbb{P}^2$ through r. What is that point r?

2. Now let C be a smooth plane curve of degree d, and C^* its dual curve in $\mathbb{P}^{2^*}$. To determine the degree of C^*, it is enough (by the Bezout or Baby Bezout theorem) to intersect C^* with a generic line M in $\mathbb{P}^{2^*}$ and count the number of points.

Considering the previous question, and the definition of the dual curve, explain why this is the same as picking a general point r of $\mathbb{P}^2$ and asking how many lines L in $\mathbb{P}^2$ passing through r are tangent to C.

3. Consider projecting C from r onto some complementary line L in $\mathbb{P}^2$. The ramification points of this map $\pi : C \longrightarrow L$ are exactly the points p of C such that the tangent line L_p to C at p passes through r. By choosing the point r generically enough, you may assume that all the ramification points are simple (i.e., that the ramification index $k_p = 2$ for all ramification points p of the map π).

Considering the map $\pi : C \longrightarrow L \cong \mathbb{P}^1$ as a map of Riemann-Surfaces, and assuming that C is a degree d plane curve, use the Riemann-Hurwitz formula to compute the number of ramification points, and hence the degree of C^*. (As a check, if $d = 2$, C^* will also be of degree 2. If $d = 3$, C^* will be of degree 6.)

4. We now want to show that $(C^*)^* = C$. There is obviously something a bit suspicious about this: if you iterate the formula from the previous question, the degree in general goes up (except for $d = 2$), and so this statement seems impossible. The reason is that the curve C^* is unfortunately a singular curve, and there are a finite number of points of C^* where C^* does not have a well defined tangent line, so the previous formula is not valid when applied to C^*.

This also means we should be a bit more careful on how we're defining the dual $(C^*)^*$ of C^*. In order to do this, we restrict ourselves to points q of C^* where C^* does have a tangent line, and define the point in $\mathbb{P}^2$ of $(C^*)^*$ corresponding to $q \in C^*$ to be the point of $\mathbb{P}^2$ corresponding to the tangent line M_q to C^* at q (the image of the dual of C^* for the bad points can then be computed by continuity, or by the "extension lemma" from class). The only way that this extra wrinkle will effect us is that in the following computation we will be allowed to ignore finitely many points where something bad or inconvenient happens.

To begin the computation, let p be a point of C, and U a small chart of C around p. On this chart U, we can assume that the inclusion $C \hookrightarrow \mathbb{P}^2$ is given by

$$q \in U \mapsto [f_1(q) : f_2(q) : f_3(q)]$$

for some holomorphic functions f_1, f_2, and f_3 defined on U.

Explain why the tangent line $aX + bY + cZ = 0$ to C at p will be given by $[a : b : c]$ such that $af_1(p) + bf_2(p) + cf_3(p) = 0$ and $af_1'(p) + bf_2'(p) + cf_3'(p) = 0$, where f_i' denotes the derivative of f_i on U (with respect to some chosen coordinate on U).

5. This is the same as saying that $[a, b, c]$ is in the kernel of the linear map $\mathbb{C}^3 \longrightarrow \mathbb{C}^2$ given by the matrix

$$\begin{bmatrix} f_1(p) & f_2(p) & f_3(p) \\ f_1'(p) & f_2'(p) & f_3'(p) \end{bmatrix}.$$

Explain why the collection

$$[(f_2(p)f_3'(p) - f_2'(p)f_3(p)) : -(f_1(p)f_3'(p) - f_1'(p)f_3(p)) : (f_1(p)f_2'(p) - f_1'(p)f_2(p))]$$

of 2×2 minors of this matrix is in the kernel. This argument is valid for any $q \in U$, so that on U we may take the map $U \longrightarrow C^* \subset \mathbb{P}^{2^*}$ to be given by

$$q \mapsto [(f_2(q)f_3'(q) - f_2'(q)f_3(q)) : -(f_1(q)f_3'(q) - f_1'(q)f_3(q)) : (f_1(q)f_2'(q) - f_1'(q)f_2(q))].$$

6. Defining h_1, h_2, h_3 to be the minors as above, we now have a description of the map $U \longrightarrow \mathbb{P}^{2^*}$ in terms of three holomorphic functions:

$$q \mapsto [h_1(q) : h_2(q) : h_3(q)].$$

Repeat the argument in questions 4 and 5 to compute the formula for the tangent line at q to C^*, giving the map back to $\mathbb{P}^2$. Writing out the computation in terms of the functions f_i, show that (as long as a certain matrix determinant involving f_i, f_i', and f_i'' doesn't vanish) that this double dual map is exactly

$$q \mapsto [f_1(q) : f_2(q) : f_3(q)]$$

again, showing that $(C^*)^* = C$. (The matrix determinant only vanishes at flex points of C, so except for finitely many points of C the computation is valid. The flex points can then be taken care of by continuity, an argument which you may omit.)

IV. — QUOTIENTS OF AFFINE CURVES BY FINITE GROUPS.

INTRODUCTION

Suppose that we are working in some fixed geometric category, for instance the world of differential manifolds, or complex analytic manifolds, or the algebraic world. Suppose we have an object X of that category (i.e., X is a C^∞ manifold, complex analytic manifold, etc.) and that we have a finite group G acting on X. We want to understand what it means to take the quotient of X by G. That is, we seek an object Y in our world such that "the points of Y correspond to the G-orbits in X".

There is an elegant way to describe what we want. Call a map $\phi : X \longrightarrow W$ an *orbit-map* if, for all $q \in W$ the inverse image $\phi^{-1}(q)$ is a union of orbits, i.e., if $p \in \phi^{-1}(q)$ then $g \cdot p \in \phi^{-1}(q)$ for all $g \in G$.

We call an object Y in our geometric world the *quotient of X by G* (symbol: $X /\!\!/ G$) if there is an orbit map $\pi : X \longrightarrow Y$ which is *universal* for all orbit maps in our geometric category. That is, if W is an object in our category, and $\phi : X \longrightarrow W$ is an orbit map, then there is a unique $\psi : Y \longrightarrow W$ so that $\phi = \psi \circ \pi$.

$$\begin{tikzcd} X \arrow[r, "\pi"] \arrow[dr, "\phi"'] & Y \arrow[d, dotted, "\exists\psi"] \\ & W \end{tikzcd}$$

This definition may seem difficult to verify, but there are many situations where the construction of Y can be made very explicit. One situation is trying to take the quotient of an affine curve by a G-action. The reason is that associated to an affine curve is its ring of algebraic functions, a ring from which it's possible to recover everything about the affine curve. Thinking about what the definition above means in terms of rings makes it possible to construct the quotient Y directly.

More generally, there are geometric objects called *affine varieties* which are also completely determined by, and can be reconstructed from, their rings of algebraic functions. The argument below works unchanged in the generality of affine varieties. However, since we only understand the case of affine curves, that is the only case we will consider below.

PURPOSE

The purpose of this question is to use the somewhat esoteric sounding correspondence between affine curves and their rings of functions to work with the seemingly abstract definition above; hopefully this will make both of them seem more concrete. It is also a chance to think about the meaning of "pullback of functions" and the formalism associated with the correspondence between rings and geometric objects.

NOTATION

Because we will be dealing with both geometric objects and their rings of functions, we will use letters at the end of the alphabet (X, Y, W) for the geometric objects, and letters at the beginning (A, B, C) for rings. In particular, in this question the letter C does *not* refer to a curve.

SETUP

Suppose that X is an *affine curve*. For us that means that X is given by an algebraic equation $f(x, y) = 0$ in $\mathbb{C}^2$, the function f satisfying the necessary derivative conditions to make X a smooth manifold. More generally, we could consider the zero locus $f_1 = f_2 = \cdots = f_r = 0$ in some $\mathbb{C}^k$, with sufficient conditions on the f_i's to guarantee that the zero locus was a smooth 1-dimensional manifold. Geometrically X is a projective curve (= Riemann surface), minus a finite number of points.

The ring of algebraic functions A on X is by definition the restriction of the polynomial functions on $\mathbb{C}^2$ to X, or equivalently the quotient ring $\mathbb{C}[x, y]/(f)$. In the more general case mentioned above, the ring of polynomial functions is (for the same reasons) $\mathbb{C}[x_1, \ldots, x_k]/(f_1, \ldots, f_r)$.

Suppose that X is an affine curve, with ring of algebraic functions A, and that W is an affine curve with ring of algebraic functions C. Given an algebraic map $\phi : X \longrightarrow W$, then for any function $f \in C$, we can pull back that function to a function $\phi^*(f)$ on X by

$$(\phi^*(f))(p) := f(\phi(p)) \ \ \forall p \in X.$$

The resulting function $\phi^*(f)$ is an algebraic function on X, and hence an element of the ring A. This procedure gives rise to a *ring homomorphism*

$$\phi^* : C \longrightarrow A$$

in the "opposite" direction to the map ϕ. (i.e., pullback is a contravariant functor.)

The fundamental fact about affine curves (and affine varieties in general) is that *every* ring homomorphism $C \longrightarrow A$ arises uniquely as the pullback ϕ^* for some algebraic map $\phi : X \longrightarrow W$. In other words, we have

$$(*) \qquad \{\text{Algebraic maps } \phi : X \longrightarrow W\} \overset{\text{1-1}}{\longleftrightarrow} \left\{ \begin{array}{c} \text{Ring homomorphisms} \\ \phi^* : C \longrightarrow A \end{array} \right\}.$$

Suppose that we have an affine curve X, and a finite group G acting (on the left) on X. For convenience we assume that G acts *faithfully*, that is, the only element acting trivially on all of X is the identity element of G. (if G is not acting faithfully, we can always mod out by the kernel of the action to reduce to this case). We want to try and understand or construct the quotient $X/\!\!/G$.

Problems

1. If G acts on X on the left, explain why the ring A of algebraic functions on X has a natural *right* G-action. (This is sometimes made into a left action on A by declaring the left action by g to be the natural right action by g^{-1}.)

If $f \in A$ is a function on X, $g \in G$ an element of the group, and p a point of X, what is the value of $f \cdot g$ at p? (Where $f \cdot g$ means the right action of g on f.)

2. If W is an affine curve, with ring of algebraic functions C, and $\phi : X \longrightarrow W$ is an orbit map show that for any function $f \in C$, any $g \in G$ and any $p \in X$, we have $(\phi^*(f))(p) = (\phi^*(f))(g \cdot p)$.

3. If $B := A^G$ is the subring of A of functions invariant under G, and W is an affine curve with ring of functions C, explain why a map $\phi : X \longrightarrow W$ is an orbit map if and only if the image of C under ϕ^* lies in B.

In solving this problem you may use the following fact: If q and q' are distinct points of W (i.e., $q \neq q'$) then there is some function $f \in C$ such that $f(q) \neq f(q')$.

4. If Y is an affine curve with ring of algebraic functions B ($B = A^G$ as before), and we define the map $\pi : X \longrightarrow Y$ to be the one corresponding via (*) to the natural inclusion $B \hookrightarrow A$, explain why Y has the desired universal property in the world of affine curves.

5. Thanks to the previous question, given an affine curve X with action by a finite group G, constructing the quotient amounts to finding the invariant ring $B = A^G$ and finding the affine curve Y associated with that ring.

Carry this procedure out in the following concrete case: Let X be the curve in $\mathbb{C}^2$ cut out by $f(x, y) = y^2 - x^3 + x = 0$. Let G be the cyclic group of order 2 where for any point $(x, y) \in \mathbb{C}^2$ on X the nontrivial element of G acts by $(x, y) \mapsto (x, -y)$.

Verify that this is an action of G on X, find the invariant ring $B = A^G$, and find an affine curve Y which has that ring of functions. Also, identify geometrically the map $\pi : X \longrightarrow Y$ corresponding to the inclusion $B \hookrightarrow A$.

6. If you know more commutative algebra, try and answer this question: In general, what is the degree of the map $\pi : X \longrightarrow Y$? What (in terms of the group action) are the ramification points and ramification indices of the map π? Can you prove this?

V. — A DESCRIPTION OF THE PICARD GROUP BY GENERATORS AND RELATIONS.

INTRODUCTION

The *Picard group* $\mathrm{Pic}(C)$ of a curve C is the group of line bundles on C, the group operation, identity, and inverse being the ones described in class. This group is an important object attached to the curve C, although it may perhaps seem too abstract to work with.

There is a second classical description of the Picard group, "Divisors modulo linear equivalence", given in terms of generators and relations which is sometimes used as the definition. It has the advantage of appearing to be more concrete, although since there are an uncountable number of generators, and an uncountable number of relations, it also has its difficulties.

This description may also be more familiar if you've studied some commutative algebra or number theory. From the point of view of generators and relations, it will be clear that, in the algebraic setting, the Picard group is the same as the class group.

PURPOSE

The idea of this question is to derive and justify this second description of the Picard group.

DEFINITIONS AND NOTATION

Fix a smooth curve C.

The *group of divisors* $\mathrm{Div}(C)$ on C is the free abelian group with basis the points p of C. An element D of $\mathrm{Div}(C)$, called a *divisor*, is therefore a formal sum

$$D = \sum_{p \in C} n_p \cdot p$$

where the n_p are integers, zero except for finitely many $p \in C$.

We also want to look at the set of pairs of a line bundle with meromorphic section, the section being defined only up to scalar multiple:

$$\mathrm{LS}(C) := \left\{ (\mathcal{L}, s) \;\middle|\; \begin{array}{l} \mathcal{L} \text{ is a line bundle on } C, \text{ and } s \text{ is} \\ \text{nonzero meromorphic section of } \mathcal{L} \end{array} \right\} / \sim .$$

where we divide out by the relation $(\mathcal{L}, s) \sim (\mathcal{L}, \lambda s)$ for any $\lambda \in \mathbb{C}, \lambda \neq 0$.

If the line bundle $\mathcal{L}$ is given in terms of transition functions $\{\tau_{ij}\}$ relative to some open cover $\{U_i\}$ of C, then we can describe a meromorphic section s by a collection of functions f_i, meromorphic on each U_i, such that $f_j = \tau_{ij} f_i$ on $U_i \cap U_j$.

The set $\mathrm{LS}(C)$ is actually also a group, the group operation in the "$\mathcal{L}$" coordinate being the usual operation with line bundles, and in the "s" coordinate multiplication of sections.

More concretely, suppose that we have two line bundles $\mathcal{L}$ and $\mathcal{L}'$ with meromorphic sections s and s', and that we choose an open cover $\{U_i\}$ of C such that both $\mathcal{L}$ and $\mathcal{L}'$ can be described by transition functions $\{\tau_{ij}\}$ and $\{\tau'_{ij}\}$ with respect to this cover. We can also describe the meromorphic sections using functions $\{f_i\}$ and $\{f'_i\}$ with respect to this cover, so that we have:

Line bundle	transition functions	section	local description of section
$\mathcal{L}$	$\{\tau_{ij}\}$	s	$\{f_i\}$
$\mathcal{L}'$	$\{\tau'_{ij}\}$	s'	$\{f'_i\}$

Then the group structure is given by

	Line bundle	transition functions	section	local description
Mult.	$\mathcal{L} \otimes \mathcal{L}'$	$\{\tau_{ij}\tau'_{ij}\}$	ss'	$\{f_i f'_i\}$
Inv.	$\mathcal{L}^*$	$\{1/\tau_{ij}\}$	$1/s$	$\{1/f_i\}$
Id.	$\mathcal{O}_C$	$\{1\}$	"1"	$\{1\}$

That is, if $(\mathcal{L}, s)$ and $(\mathcal{L}', s')$ are two elements of $\mathrm{LS}(C)$, their product in $\mathrm{LS}(C)$ is $(\mathcal{L} \otimes \mathcal{L}', ss')$, and the inverse of $(\mathcal{L}, s)$ is $(\mathcal{L}^*, 1/s)$, where the meanings of ss' and $1/s$ are described by the table above.

If $(\mathcal{L}, s)$ is an element of $\mathrm{LS}(C)$, then we can define an element $\mathrm{div}(s)$ of $\mathrm{Div}(C)$ as follows: for any point p of C, let n_p be the order of zero or pole (poles counted negatively) of s at p, i.e., the order of f_i at p if $p \in U_i$. Since a meromorphic section only has finitely many zeros or poles, $n_p = 0$ for all but finitely many p, so that $\mathrm{div}(s) := \sum_p n_p \cdot p$ is an element of $\mathrm{Div}(C)$.

The construction "div" gives a map $\mathrm{div} : \mathrm{LS}(C) \longrightarrow \mathrm{Div}(C)$.

In answering the questions, you may assume the following fact:

(*) If $(\mathcal{L}, s)$ is an element of $\mathrm{LS}(C)$, and s has no zeros or poles anywhere, then $\mathcal{L} \cong \mathcal{O}_C$ and s is a scalar multiple of the section 1, that is, $(\mathcal{L}, s)$ is the identity element of $\mathrm{LS}(C)$.

PROBLEMS

1. Show that $\mathrm{div} : \mathrm{LS}(C) \longrightarrow \mathrm{Div}(C)$ is an injective group homomorphism.

2. Explain why that in order to show that div is also surjective, it is sufficient to show that for any point $p \in C$ there is a line bundle $\mathcal{L}$ on C with global section s such that $\mathrm{div}(s) = 1 \cdot p$.

3. Given a $p \in C$ construct such a pair $(\mathcal{L}, s)$ as follows: Pick a small chart U_1 around p with coordinate z on U_1 so that $z = 0 \leftrightarrow p$. Let $U_2 = C \setminus \{p\}$, and define a line bundle $\mathcal{L}$ by the transition functions $\tau_{12} = 1/z$, $\tau_{21} = z$. Define a section s of $\mathcal{L}$ by the functions $f_1 = z$ on U_1, $f_2 = 1$ on U_2. Check that this does define a section, and that $\operatorname{div}(s) = 1 \cdot p$.

4. We also have a map $\mathrm{for} : \mathrm{LS}(C) \longrightarrow \mathrm{Pic}(C)$ forgetting the meromorphic section, i.e., that takes an element $(\mathcal{L}, s)$ of $\mathrm{LS}(C)$ to the line bundle $\mathcal{L} \in \mathrm{Pic}(C)$. This is a group homomorphism by our definition of the group operations on $\mathrm{LS}(C)$.

This gives us the following diagram:

$$\begin{tikzcd} \mathrm{LS}(C) \arrow[r, "\overset{\mathrm{div}}{\sim}"] \arrow[d, "\mathrm{for}"'] & \mathrm{Div}(C) \arrow[d] \\ \mathrm{Pic}(C) \arrow[r, "\sim"] & ?? \end{tikzcd}$$

where the map div is an isomorphism (by questions 1, 2, and 3) and the map for is a surjection, since any line bundle $\mathcal{L} \in \mathrm{Pic}(C)$ has a meromorphic section s.

Since for is a surjection, $\mathrm{Pic}(C)$ is a quotient of $\mathrm{LS}(C)$. Since div is an isomorphism, $\mathrm{Pic}(C)$ is therefore also a quotient of $\mathrm{Div}(C)$. Our description of $\mathrm{Pic}(C)$ by generators and relations will be complete once we figure out an explicit description of the subgroup of $\mathrm{Div}(C)$ we should quotient out by in order to get $\mathrm{Pic}(C)$.

Identify the kernel of the map for, and then apply div to that to show that the group we should divide out by is those divisors of the form $\operatorname{div}(h)$ where h is a meromorphic *function* on C.

5. Define an equivalence relation $\sim$ on $\mathrm{Div}(C)$ as follows: for any two divisors D, D' of $\mathrm{Div}(C)$, $D \sim D'$ if and only if there is a meromorphic function h on C with

$$D - D' = \operatorname{div}(h),$$

where $D - D'$ is subtraction in $\mathrm{Div}(C)$.

Show that this does define an equivalence relation, and explain why $\mathrm{Pic}(C) = \mathrm{Div}(C)/\sim$.

VI. — The genus of a curve on $\mathbb{P}^1 \times \mathbb{P}^1$.

Introduction

On the surface $\mathbb{P}^2$, the essential algebraic number describing the equation of a plane curve is its degree d. The essential geometric number attached to the curve is its genus g. On $\mathbb{P}^2$ these are related by the formula $g = \binom{d-1}{2}$.

Purpose

The purpose of this question is to work out the corresponding formula for the surface $\mathbb{P}^1 \times \mathbb{P}^1$. Here however each curve on $\mathbb{P}^1 \times \mathbb{P}^1$ comes with a *pair* (d, e) of numbers to describe it algebraically, called the *bidegree* of the curve.

Definitions and Notation

The product $\mathbb{P}^1 \times \mathbb{P}^1$ consists of the pairs of points

$$\mathbb{P}^1 \times \mathbb{P}^1 = \left\{ (u, v) \mid u \in \mathbb{P}^1, v \in \mathbb{P}^1 \right\}.$$

We can represent a point of $\mathbb{P}^1 \times \mathbb{P}^1$ by a pair of homogeneous coordinates

$$(u, v) = ([X_0 : X_1], [Y_0, Y_1]),$$

where of course the homogeneous coordinates are only defined up to scaling:

$$([X_0 : X_1], [Y_0, Y_1]) \sim ([\lambda X_0 : \lambda X_1], [\mu Y_0 : \mu Y_1]),$$

for $\lambda, \mu \in \mathbb{C}$, $\lambda \neq 0$, $\mu \neq 0$. Note that we can scale the coordinates separately.

This set is also a complex manifold. For instance, we can cover $\mathbb{P}^1$ by two charts, each isomorphic to $\mathbb{C}$, and so it follows that we can cover $\mathbb{P}^1 \times \mathbb{P}^1$ by the four possible product charts $\mathbb{C} \times \mathbb{C} \cong \mathbb{C}^2$.

A curve $C \subset \mathbb{P}^1 \times \mathbb{P}^1$ is called *smooth* if, in each of the four $\mathbb{C}^2$ charts it is given as the zero locus of a holomorphic function $f(x, y) = 0$ satisfying the usual derivative conditions. This is enough to imply that C is itself a smooth compact 1-dimensional complex manifold, i.e. what we usually mean by a curve or Riemann surface.

A polynomial

$$F = \sum_{i,j,k,l} a_{ijkl} X_0^i X_1^j Y_0^k Y_1^l$$

is said to be *bihomogeneous of bidegree* (d, e) if for every nonzero monomial $X_0^i X_1^j Y_0^k Y_1^l$ we have $i + j = d$ and $k + l = e$. In other words, the "X-degree" of each monomial is d and the "Y-degree" of each monomial is e. As an example,

$$F = X_0^4 X_1 Y_1^3 - 13 X_0^2 X_1^3 Y_0^2 Y_1 + 20 X_0^2 X_1^3 Y_0 Y_1^2 - X_1^5 Y_1^3$$

is bihomogeneous of bidegree $(5, 3)$.

PROBLEMS

1. Explain why a bihomogeneous polynomial of type (d, e) gives a well defined zero locus in $\mathbb{P}^1 \times \mathbb{P}^1$.

2. There is a natural projection map

$$\begin{array}{rcl} \mathrm{pr}_2 : \mathbb{P}^1 \times \mathbb{P}^1 & \longrightarrow & \mathbb{P}^1 \\ ([X_0 : X_1], [Y_0 : Y_1]) & \mapsto & [Y_0 : Y_1] \end{array}$$

projecting onto the second set of homogeneous coordinates. (There is also a projection map onto the first set of coordinates, but we'll only need one of these.)

Suppose that $C \subset \mathbb{P}^1 \times \mathbb{P}^1$ is a smooth curve given by a bihomogeneous equation $F = 0$ of bidegree (d, e). Applying the map pr_2 to C, we get a map $\pi : C \longrightarrow \mathbb{P}^1$.

In terms of the equation $F = 0$ for C, for any point $[\alpha : \beta] \in \mathbb{P}^1$, how can you compute the preimage $\pi^{-1}([\alpha : \beta])$? What is the degree of the map π?

3. Plugging in $Y_0 = \alpha$, $Y_1 = \beta$ into F, we get a homogeneous polynomial

$$F_{\alpha,\beta} = \sum_{i+j=d} a_{ij} X_0^i X_1^j$$

in two variables which therefore must factor as a direct product of linear factors

$$F_{\alpha,\beta} = \prod_{s=1,\dots,r} (b_s X_0 - a_s X_1)^{m_s},$$

possibly with some multiplicities.

Explain why $[\alpha : \beta]$ is a branch point of π if and only if the polynomial $F_{\alpha,\beta}$ has repeated roots. In terms of the factorization of $F_{\alpha,\beta}$, how can you read off the ramification points of π and their ramification indices?

4. In order to apply Riemann-Hurwitz to C, we need to understand the ramification and branching data of π. Let us consider the problem of detecting when a homogeneous polynomial of the form $G = \sum_{i+j=d} a_{ij} X_0^i X_1^j$ has multiple roots.

Explain why, if G has a multiple root then G and $\frac{\partial}{\partial X_0} G$ will have roots in common. Explain why, conversely, if X_1 doesn't divide G (i.e., $a_{d0} \neq 0$) and G and $\frac{\partial}{\partial X_0} G$ have a root in common, that G has a multiple root.

5. Considering the polynomial F as a homogeneous polynomial of degree d in X_0 and X_1, with coefficients homogeneous polynomials in Y_0 and Y_1, what is the degree of the resultant of F and $\frac{\partial}{\partial X_0} F$ as a polynomial in Y_0 and Y_1?

6. The degree above counts not only the branch points on $\mathbb{P}^1$ of the map π, but also (in light of question 4) those points $[Y_0 : Y_1]$ of $\mathbb{P}^1$ where the coefficient a_{d0} of $F_{\alpha,\beta}$ vanishes. What is the degree of the coefficient of X_0^d in F, as a function of Y_0 and Y_1? How many times on $\mathbb{P}^1$ is this extra vanishing therefore expected to happen?

Subtracting this number from the number in question 5, how many branch points is π expected to have?

7. Under the following assumptions: (i) all the ramification points of π are simple (i.e. $k_p = 2$ for all ramification points), (ii) the ramification points of π map to distinct branch points, (iii) the number of branch points is that predicted by the number from question 6, use the Riemann-Hurwitz formula to compute the genus of C.

As a check: the formula for the genus should be symmetric in d and e, and, if either d or e is 1, the genus should be zero. (These statements should be clear geometrically after a little thought).

8. According to your formula, do all values of $g \geq 0$ seem possible for smooth curves on $\mathbb{P}^1 \times \mathbb{P}^1$? What about for smooth curves on $\mathbb{P}^2$?

VII. — LINE BUNDLES ON $\mathbb{P}^2$.

PURPOSE

To practice computing with line bundles using the transition function definition.

NOTATION AND DEFINITIONS

We use the three usual charts for $\mathbb{P}^2$:

Name of Chart	Description	Coordinates
U_1	$Z \neq 0$	$x_1 = \frac{X}{Z}$, $y_1 = \frac{Y}{Z}$
U_2	$X \neq 0$	$y_2 = \frac{Y}{X}$, $z_2 = \frac{Z}{X}$
U_3	$Y \neq 0$	$x_3 = \frac{X}{Y}$, $z_3 = \frac{Z}{Y}$

On the intersections $U_i \cap U_j$ the functions are related by

$U_1 \cap U_2 :$	$x_1 = \dfrac{1}{z_2}$, $y_1 = \dfrac{y_2}{z_2}$ and $y_2 = \dfrac{y_1}{x_1}$, $z_2 = \dfrac{1}{x_1}$
$U_2 \cap U_3 :$	$y_2 = \dfrac{1}{x_3}$, $z_2 = \dfrac{z_3}{x_3}$ and $x_3 = \dfrac{1}{y_2}$, $z_3 = \dfrac{z_2}{y_2}$
$U_1 \cap U_3 :$	$x_1 = \dfrac{x_3}{z_3}$, $y_1 = \dfrac{1}{z_3}$ and $x_3 = \dfrac{x_1}{y_1}$, $z_3 = \dfrac{1}{y_1}$

For any integer d (positive or negative) we define the line bundle $\mathcal{O}_{\mathbb{P}^2}(d)$ (i.e., $\mathcal{O}_{\mathbb{P}^2}(d)$ is the *name* of this line bundle, instead of L, or some other symbol) by the transition functions:

$$\tau_{12} = \frac{1}{x_1^d}, \qquad \tau_{21} = \frac{1}{z_2^d},$$
$$\tau_{23} = \frac{1}{y_2^d}, \qquad \tau_{32} = \frac{1}{x_3^d},$$
$$\tau_{13} = \frac{1}{y_1^d}, \qquad \tau_{31} = \frac{1}{z_3^d}.$$

PROBLEMS

1. Verify that the transition functions $\{\tau_{ij}\}$ for $\mathcal{O}_{\mathbb{P}^2}(d)$ satisfy the cocycle conditions.

2. Verify that the functions

$$\begin{aligned} f_1(x_1, y_1) &= x_1^4 - x_1^2 y_1^3 + 3x_1 y_1 - 2 \\ f_2(y_2, z_2) &= z_2 - y_2^3 + 3y_2 z_2^3 - 2z_2^5 \\ f_3(x_3, z_3) &= x_3^4 z_3 - x_3^2 + 3x_3 z_3^3 - 2z_3^5 \end{aligned}$$

on U_1, U_2, and U_3 satisfy the compatibility conditions to give a global section of $\mathcal{O}_{\mathbb{P}^2}(5)$.

3. In general, for $d \geq 0$ and holomorphic functions

$$\begin{aligned} f_1(x_1, y_1) &= \sum_{i,j \geq 0} a_{ij} x_1^i y_1^j \\ f_2(y_2, z_2) &= \sum_{i,j \geq 0} b_{ij} y_2^i z_2^j \\ f_3(x_3, z_3) &= \sum_{i,j \geq 0} c_{ij} x_3^i z_3^j \end{aligned}$$

on the charts U_1, U_2, and U_3, write down the compatibility conditions on the coefficients for the f_i's to give a global section of the bundle $\mathcal{O}_{\mathbb{P}^2}(d)$.

In light of the GAGA principle, even though we start with arbitrary holomorphic functions, the compatibility conditions should force the f_i's to be polynomials.

4. Using the conditions from the previous question, show that for $d \geq 0$ the global sections of $\mathcal{O}_{\mathbb{P}^2}(d)$ on $\mathbb{P}^2$ are in one to one correspondence with the homogeneous polynomials of degree d in X, Y, and Z. (You may also want to compare question 2 above with question 1 from Homework assignment 3.)

5. In terms of d ($d \geq 0$), give a formula for $h^0(\mathcal{O}_{\mathbb{P}^2}(d))$, the dimension of the vector space of global sections of $\mathcal{O}_{\mathbb{P}^2}(d)$.

6. Complete the analysis of the line bundles $\mathcal{O}_{\mathbb{P}^2}(d)$ by showing that for $d < 0$, $\mathcal{O}_{\mathbb{P}^2}(d)$ has no (nonzero) global sections.